

Simultaneous Search with Heterogeneous Firms and Ex Post Competition*

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Abstract

It is well known that in economies with search frictions high and low productivity firms can coexist. In this paper we are interested in whether workers select their portfolios of applications to different firm types in a socially desirable way. Specifically, we study a search model where workers can apply to high and or low productivity firms. Firms that compete for the same candidate can increase their wage offers as often as they like. We show that if workers apply to two jobs, there is a unique symmetric equilibrium where workers mix between sending both applications to the high and sending both to the low productivity sector. But, efficiency requires that they apply to both sectors because a higher matching rate in the high-productivity sector can then be realized with fewer applications (and consequently fewer coordination frictions) if workers always accept the offer of the most productive sector. However, in the market the worker's payoff is determined by how much the firm with the second highest productivity is willing to bid. This is what prevents them from applying to both sectors. For many configurations, the equilibrium outcomes are the same under directed and random search. Allowing for free entry creates a second source of inefficiency. We discuss the effects of increasing the number of applications and show that our results can easily be generalized to N -firms.

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1 Introduction

In most labor markets, heterogeneous firms compete for workers. In a competitive environment more productive firms can always bid more than less productive firms so they will hire all workers and ultimately the unproductive firms are driven out of the market. In a world with frictions, high and low productive firms can coexist but the question whether the allocation of workers over both types of firms is optimal given the frictions remains. In this paper we show that under arguably small coordination frictions the optimal allocation breaks down. We consider the following deviations from the competitive model: (i) workers do not know to which firms other workers apply to, (ii) firms do not know which candidates receive offers, (iii) applications are costly and firms can consider only a fraction of their candidates. While keeping our model as simple as possible we want to capture a number of factors that we feel are important in real world labor markets like heterogeneity, the possibility of simultaneous search and ex post competition for workers with multiple offers. At the same time we want to rigorously model the matching process, the strategic interactions between workers with each other and with the firms.

Specifically, we study a portfolio problem where identical unemployed workers must decide in which sector(s) to search, the high and or the low productivity sector. Within a sector, all firms are identical. Workers can send 0, 1 or 2 applications at a cost $k > 0$ for each application. Each vacancy that receives one or more candidates randomly picks a candidate and offers the job to him. The other applications are rejected. In the simplest version of the model, workers know the productivity in each sector but only learn about the wage at a specific firm after applying there. We then show that our results still hold in the much more complicated case where search is fully directed: i.e. firms can ex ante post a wage which is observed by all workers before they decide where to send their applications. Firms that compete for the same candidate can increase their offers as often as they like so we do not restrict the firm's strategy space in this dimension. We are interested in symmetric pure strategy equilibria (in terms of the number of applications) and their efficiency properties. Interestingly, in the simplest version of our model it cannot be an equilibrium for workers to send just one application because then firms have no incentives to offer a positive wage. This is basically the Diamond (1971) paradox. Therefore, if k is sufficiently low, workers always send two applications, hoping to get a positive payoff by receiving two offers. But this in turn implies that workers will never apply to both sectors (HL) because this strategy is strictly dominated by sending both applications to the low productivity sector (LL). The intuition behind this result is that in any equilibrium where workers are willing to apply to the low productivity sector, the expected number of applications must be lower there. However, the expected payoffs of receiving an offer from a high and a low productivity firm is the same as receiving offers from two low productivity firms because a high productivity firm that (Bertrand) competes with a low productivity firm for the same candidate will win and pay the productivity level of the worker at the low productivity firm. So, the worker's payoffs conditional on getting two offers are the same for a worker who sends both applications to the low productivity sector (LL) and a worker who plays HL , but the probability of receiving two offers is higher for the first worker. We then show that there is a unique mixed

strategy equilibrium where workers send both applications with probability q_{HH}^* to the high productivity sector and with probability $1 - q_{HH}^*$ to the low productivity sector where q_{HH}^* depends on the relative productivity and the relative supply of vacancies in each of the sectors. As in Albrecht et al. (2006) there are two coordination problems in the matching process: (1) workers do not know where other workers apply to and (2) firms do not know which candidate other firms consider.

By allowing workers to apply to different sectors, the degree of coordination frictions becomes partly endogenous, even for a given number of applications per worker. However, workers do not internalize the effects of their portfolio choice on the employment opportunities of other workers. They just want to maximize the productivity-weighted probability to receive multiple offers. We show that the resulting equilibrium is not efficient and unemployment is too high. An important reason for the inefficiency is that a social planner would like some or all workers to apply to both sectors in order to reduce the coordination problems in the matching process. More H matches can be realized by letting workers accept the job in the most productive sector in case of multiple offers. In the market, workers never play HL because the expected payoffs of this strategy are too low, since high productivity firms would either pay the monopsony wage or the productivity level of a low productivity firm in case the worker has two offers. Since the expected payoff of playing HL is independent of high productivity output, workers incentives are distorted. Another source of inefficiency is that because of the coordination frictions, the matching function is non-monotonic in the number of applications. When there are relatively few vacancies, the second coordination problem is severe and the matching rate is decreasing in the number of applications. The planner internalizes this while individual workers apply too often to the high productivity sector. A similar problem arises in the academic job market or the market for Ph.D. candidates where the top universities typically receive too many applicants.¹

If the number of firms in the market or the difference in productivity between both sectors is not too large, the equilibrium outcomes under random search are the same as in the directed search equilibrium where firms can post a wage ex ante and workers observe all wages.² The reason for this is the same as the one in Albrecht et al. (2006) where posted wages are zero. They consider the case where all workers and firms are identical and show that the existence of ex post competition makes it still attractive for workers to apply to firms who offer the monopsony wage. Offering a higher wage than the monopsony wage only marginally increases the number of applicants in expectation, because workers mainly care about the probability to get multiple offers, while the expected firm payoffs in case of a match drop linearly.

In section 3 we also allow for free entry of vacancies. We do this by allowing the output of both sectors to be traded in a competitive goods market where consumers with love-for-variety demand both types of goods.³ Now, not only the workers' incentives are distorted, but also firms' incentives are distorted. Vacancy supply in each sector can both be too high or too low while typically, the market assigns too few workers to

¹In small labor markets, more matches are realized if all workers play HL than if 50% plays LL and 50% plays HH. However, in large labor markets there is no difference between these two cases.

²Usually, the equilibrium in directed search models is constrained efficient, e.g. Burdett, Shi and Wright (2001), Moen (1999), Montgomery (1991), Peters (1991).

³The fixed vacancy supply case can be considered to be a special case with Leontief demand. Further, if output in both sectors are perfect substitutes, only one good will be produced namely the one where the expected value of a vacancy is highest.

the high productivity sector. Even if we restrict the planner to playing only HH and LL , the inefficiency remains.

There are a couple of papers related to what we do. First, Shimer (2005) and Shi (2002) consider a directed search model with two-sided heterogeneity where workers can only apply to one job and ex post competition is irrelevant. They find that the decentralized market outcome is constrained efficient. We show that this result may break down if workers can simultaneously apply to multiple jobs. Our model reduces to Albrecht et al. (2006) when both sectors have the same productivity. Then, workers randomize between all firms and the possibility of ex post Bertrand competition drives down the ex ante posted wages to zero. In Gautier and Moraga-Gonzalez (2004) workers and firms are also identical and workers only learn about the wage after a firm is contacted. There, wages and the number of applications are determined in a simultaneous move game. Chade and Smith (2006) and Galenianos and Kircher (2005) also consider portfolio problems of workers who can apply to multiple jobs. In the latter paper, all jobs have the same productivity but because firms must commit to their posted wages they respond to the worker's desire to diversify. This desire to diversify is driven by the fact that the expected payoff is equal to the maximum wage offer of a worker and not to the average one. Chade and Smith (2006) is not an equilibrium model but it considers a general class of portfolio problems in the absence of ex post competition. Finally, Davis (2001) analyzes a model in which workers and firms can decide to invest in human capital and job quality respectively. Because they cannot capture the full increase of the match surplus generated by these investments, both firms and workers tend to underinvest. In equilibrium there is excessive supply of inferior jobs and inferior workers.

The paper is organized as follows. Section 2 describes the basic version of the model in which the number of vacancies is assumed to be exogenously given. We derive the equilibrium, determine whether it is efficient and furthermore check if our conclusions are sensitive to the simplifying assumptions we make. In section 3 we extend the model by allowing for free entry. Finally, section 4 concludes.

2 Basic Model

2.1 Labor Market

Consider a labor market with u risk neutral workers and v risk neutral firms. All workers are identical, but the firms are divided into two different types. There are v_H high-productivity firms and v_L low-productivity firms, with $v = v_H + v_L$. We refer to those firms as *highs* and *lows*. Each firm has exactly one vacancy.

Workers can send zero, one, or two applications at costs $k > 0$. Those applications can be directed to a specific type of vacancy, but workers do not observe ex ante the wage that a particular firm offers. If a worker receives multiple job offers, there is Bertrand competition for his services. Basically, workers must decide whether they want to send both applications to high type vacancies, both applications to low type vacancies, or one application to a high type and one to a low type vacancy. In section 2.5 we show that if there are not too many firms in the market and if the productivity of the low type firms is not too small, our results carry over to a directed search setting, where workers observe ex ante the wage offered by each

individual firm.

We make three important further assumptions. First, we assume that the labor market is large, i.e. $u \rightarrow \infty$ and $v \rightarrow \infty$, keeping $\theta_i \equiv v_i/u$ fixed $\forall i \in \{H, L\}$. For the moment, we assume that θ_H and θ_L are exogenously given. We relax this assumption in section 3. Second, we focus on symmetric equilibria, which means that identical agents must have identical strategies. This excludes equilibria that require a lot of coordination amongst workers, something that seems hard to imagine in a large labor market. Third, we assume like Shimer (2005) that the labor market is anonymous: firms must treat identical workers identically and vice versa. So, a worker's strategy may only be conditioned on the type (H or L) of the firm.

2.2 Setting of the Game

The model that is closest related to ours is the one used in Albrecht et al. (2006). There are two differences: (i) we allow for heterogeneity amongst firms and (ii) search is not fully directed. The setting of the game is as follows:

1. Each vacancy posts a wage mechanism.
2. Workers observe all vacancy types (but not the wage mechanism) and send $a \in \{0, 1, 2\}$ applications. In section 2.5 we allow workers to also ex ante observe the wage mechanism.
3. Each vacancy that receives at least one application, randomly selects a candidate. Applications that are not selected are returned as rejections.
4. A vacancy with a processed application offers the applicant the job. If the applicant receives more than one offer, the firms in question can increase their bids as often as they like.
5. A worker that receives one job offer will accept that offer as long as the offered wage is non-negative. A worker with two offers will accept the one that gives him the highest wage, or will select a job randomly if the offered wages are equal.

If a type i firm matches with a worker, it produces y_i units of output. Without loss of generality we assume that $y_L < y_H = 1$. The payoff of a firm that matches with a worker equals $y_i - w$, where w denotes the wage that the firm pays. A worker hired at wage w receives a payoff that is equal to that wage. Workers and firms that fail to match receive payoffs of zero.

2.3 Decentralized Market

We start the analysis of the decentralized market by showing that no firm posts a positive wage. This is basically the Diamond (1971) paradox.

Lemma 1 *In equilibrium all firms post a wage equal to zero*

Proof. Note that workers can direct their applications to a specific kind of vacancy, but not to a particular firm. So, posting a higher wage (or more general: a more generous wage mechanism) does not attract more applicants and does not affect the matching probability. This implies that there is no incentive for a firm to offer the worker more than zero.⁴ ■

A direct result of this lemma is that workers never send only one application.

Corollary 1 *No equilibrium exists in which there are workers that only send one application.*

Proof. Note that if a worker sends one application, there will never be ex post competition for his services. Firms offer a wage equal to zero, so the worker's payoff always equals $-k$. Hence, applying to one job is strictly dominated by not applying at all and therefore never part of an equilibrium strategy. ■

Whether a worker applies twice or not at all depends on the cost k of sending an application. For example if $k > 0.5$, each worker will decide not to apply, because applying twice costs more than the competitive wage ($2k > 1 = y_H$). On the other hand, all workers apply to two jobs if k is sufficiently small, because this gives a strictly positive expected payoff, while not applying results in a payoff of zero. In this paper we restrict ourselves to the situation in which k is small enough to guarantee that $a = 2$ with probability 1.⁵ In this respect our model differs from Shimer (2005) and Shi (2002) where $a = 1$.

Three different strategies are possible: a worker can either apply to two high type vacancies, two low type vacancies, or one high type and one low type of vacancy. Denote the respective probabilities by q_{HH} , q_{LL} , and q_{HL} , where $q_{HH} + q_{LL} + q_{HL} = 1$. Using the fact that each worker uses the same strategies, this implies that the total number of applications to firms of type i is equal to $(2q_{ii} + q_{HL})u$. The expected number of applications a specific vacancy receives, is therefore given by

$$\phi_i(q_{ii}, q_{HL}, \theta_i) = \frac{2q_{ii} + q_{HL}}{\theta_i}. \quad (1)$$

Since our labor market is large, the actual number of applications to a specific vacancy follows a Poisson distribution with mean ϕ_i .⁶ Next, consider an individual who applies to a type i firm. The number of competitors for the job at that firm also follows a Poisson distribution with mean ϕ_i , because there is an infinite number of workers. In case of n other applicants, the probability that the individual in question will get the job equals $\frac{1}{n+1}$. Therefore, the probability that an application to a type i firm results in a job offer equals

$$\begin{aligned} \psi_i &= \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{e^{-\phi_i} \phi_i^n}{n!} \\ &= \frac{1}{\phi_i} (1 - e^{-\phi_i}). \end{aligned} \quad (2)$$

⁴Note that this argument implies that posting a wage equal to zero does not only dominate posting a strictly positive wage, but also all other feasible wage mechanisms.

⁵An explicit expression for the upperbound K on k in that case is derived below.

⁶For ease of exposition we omit the arguments of functions whenever this does not lead to confusion.

Note that this expression is not well defined for $\phi_i = 0$. For convenience we define $\psi_i(0) = \lim_{\phi_i \rightarrow 0} \psi_i(\phi_i) = 1$.

Whether a worker's second application results in an offer does not depend on whether the first application was successful or not. A worker who plays ij (i.e. applies to a type i firm and a type j firm) with $i, j \in \{H, L\}$ therefore has a probability $\psi_i\psi_j$ of getting two job offers and a probability $\psi_i(1 - \psi_j) + \psi_j(1 - \psi_i)$ of getting one job offer. The matching probability of such a worker equals one minus the probability that he does not get a job offer and is therefore equal to $1 - (1 - \psi_i)(1 - \psi_j)$ (see Albrecht et al., 2006 for a proof in the case with homogenous firms). This matching probability is obviously strictly increasing in both ψ_i and ψ_j and depends on the worker's portfolio choice.

If a worker receives two high job offers, Bertrand competition between the two firms results in a wage equal to $y_H = 1$. In case of two low offers, the firms increase their bids until the worker's wage equals y_L . A combination of one high and one low offer also implies a wage of y_L , because at that wage level the low type firm is no longer willing to increase its bid. This is the standard result from Bertrand competition. As shown above, a worker who receives only one job offer gets a wage equal to zero.

Next, we prove that workers never send one application to a high firm and one to a low firm:

Lemma 2 *Workers never play HL , since this strategy is strictly dominated.*

Proof. The expected payoff for a worker who plays HL is $\psi_H\psi_L y_L - 2k$, i.e. the probability that he receives two job offers times the productivity of the low type firm minus the application cost. Likewise, the expected payoffs of playing HH and LL are $\psi_H^2 y_H - 2k$ and $\psi_L^2 y_L - 2k$ respectively. Suppose that $\psi_H \geq \psi_L$. In that case all workers play HH , since that strategy gives a strictly higher payoff than HL and LL . This however implies that $\phi_L = 0$ and thus that $\psi_L = 1$, which contradicts $\psi_H \geq \psi_L$. Hence, in equilibrium it must be the case that $\psi_L > \psi_H$. Then, playing LL gives a strictly higher payoff than HL . So, HL is strictly dominated. ■

Lemma 2 implies that there are only two potential pure strategy equilibria, one in which workers send both applications to high type firms and one in which they send both applications to low type firms. In the following proposition we show that the latter can never be an equilibrium, while the former can, but only under certain conditions.

Proposition 1 *A pure strategy equilibrium for the workers only exists if*

$$\frac{\theta_H^2}{4} \left(1 - \exp\left(-\frac{2}{\theta_H}\right) \right)^2 > y_L. \quad (3)$$

In that case $q_{HH}^ = 1$.*

Proof. There are two possibilities for a pure strategy (in terms of the sector to apply to): (i) $q_{LL} = 1$ and (ii) $q_{HH} = 1$. The case in which $q_{HL} = 1$ is ruled out by lemma 2. Since we only consider strategies in

which workers apply twice, we can safely ignore the application cost k in this proof. This parameter only plays a role in comparing the payoffs of strategies that differ in the number of applications sent.

(i) Suppose that $q_{LL} = 1$. The expected payoff for the workers then is $\psi_L^2 y_L < y_L$. A worker who deviates and applies twice to a high firm gets two high job offers and therefore a wage that equals $y_H = 1 > y_L$. So, a profitable deviation exists, which implies that $q_{LL} = 1$ is not an equilibrium.

(ii) Suppose that $q_{HH} = 1$. The expected payoff for the workers is in that case $\psi_H^2 = \frac{\theta_H^2}{4} \left(1 - \exp\left(-\frac{2}{\theta_H}\right)\right)^2$. Deviating to LL gives a wage y_L for sure. So $q_{HH}^* = 1$ is an equilibrium if condition (3) holds. ■

Hence, we have a pure strategy equilibrium in which all firms post a wage equal to zero and all workers apply twice to high type vacancies if condition (3) holds. This condition imposes very low upper bounds on y_L for any reasonable value of θ_H (e.g. $\theta_H = 0.5$ implies $y_L < 0.06$). The case in which the condition does not hold is therefore more interesting. Then, we only have a mixed strategy equilibrium.

Proposition 2 *A unique mixed strategy equilibrium exists for any $\theta_H > 0$, $\theta_L > 0$, and $y_L \in (0, 1)$ such that $\frac{\theta_H^2}{4} \left(1 - \exp\left(-\frac{2}{\theta_H}\right)\right)^2 < y_L$. This equilibrium can be characterized by the value q_{HH}^* that solves the equality $\psi_H^2 = \psi_L^2 y_L$.*

Proof. Again, we can rule out the possibility that workers play HL because of lemma 2. The only mixed strategy equilibrium that can exist is therefore one in which the workers are indifferent between playing HH and LL , i.e. where $\psi_H^2 = \psi_L^2 y_L$.

If we substitute $q_{LL} = 1 - q_{HH}$, the only free parameter in this condition is q_{HH} . To see that a unique equilibrium value q_{HH}^* exists, note that the left hand side of the condition is continuous and strictly decreasing in q_{HH} , while the right hand side is continuous and strictly increasing in q_{HH} (see Figure 1). Furthermore, we have

$$\lim_{q_{HH} \rightarrow 0} \psi_H^2 = 1 > \frac{\theta_L^2}{4} \left(1 - \exp\left(-\frac{2}{\theta_L}\right)\right)^2 y_L = \lim_{q_{HH} \rightarrow 0} \psi_L^2 y_L \quad (4)$$

and

$$\lim_{q_{HH} \rightarrow 1} \psi_H^2 = \frac{\theta_H^2}{4} \left(1 - \exp\left(-\frac{2}{\theta_H}\right)\right)^2 < y_L = \lim_{q_{HH} \rightarrow 1} \psi_L^2 y_L. \quad (5)$$

Applying the Intermediate Value Theorem now shows that there exists a unique value $0 < q_{HH}^* < 1$ such that $\psi_H^2 = \psi_L^2 y_L$ holds. ■

Unfortunately, we are not able to derive an explicit expression for q_{HH}^* . Figure 1 shows the equilibrium as the intersection point of the ψ_H^2 -curve and the $\psi_L^2 y_L$ -curve for $\theta_H = \theta_L = 0.5$ and $y_L = 0.5$. For those values 63% of the workers plays HH , while 37% plays LL .

In equilibrium the expected payoff for a worker equals $\psi_H^2 - 2k = \psi_L^2 y_L - 2k$. The requirement that this value should be larger than the payoff of not applying at all, i.e. zero, implies that k should be smaller than $\frac{1}{2}\psi_H^2 = \frac{1}{2}\psi_L^2 y_L$. This assumption seems reasonable. It is hard to imagine that the cost of a particular application exceeds half the expected wage of a job.

The equilibrium depends on three exogenous parameters, θ_H , θ_L , and y_L . The effect of a change in one of these parameters on the equilibrium values of q_{HH}^* , ϕ_i^* and ψ_i^* is summarized by the following proposition.

Proposition 3 (i) q_{HH}^* , ψ_H^* , and ψ_L^* are strictly increasing in θ_H , while ϕ_H^* and ϕ_L^* are strictly decreasing in θ_H . (ii) ψ_H^* and ψ_L^* are strictly increasing in θ_L , while q_{HH}^* , ϕ_H^* and ϕ_L^* are strictly decreasing in θ_L . (iii) ψ_H^* and ϕ_L^* are strictly increasing in y_L , while q_{HH}^* , ψ_L^* and ϕ_H^* are strictly decreasing in y_L .

Proof. See appendix. ■

This result is intuitive. A ceteris paribus increase in the number of high productivity firms increases the probability that an application to a firm of this type results in a match. Therefore, it becomes more attractive to play HH , resulting in a higher value of q_{HH}^* . The effect of the increase in the number of firms however dominates this increase in q_{HH}^* , such that the probability to get a job offer increases. Since fewer workers apply to low productivity firms, the probability to get a job offer increases there as well.

The effect of an increase in the number of low firms is similar: more workers apply to this type of vacancies and the probability to get a job offer increases at both the high and the low types of firms. A change in the productivity of the low firms does not directly affect the probability to match, but it does affect the payoff in case a worker receives two job offers from low type firms. A higher productivity of the low productivity firms is therefore associated with more applications to these firms (see also Figure 2). However, the number of low firms does not change, which implies that the probability to get a job offer decreases.

2.4 Efficiency

In the mixed strategy equilibrium that we derived in the previous subsection, a fraction q_{HH}^* of the workers matches with probability $1 - (1 - \psi_H^*)^2$ to a high firm and produce output $y_H = 1$. The remaining workers match with probability $1 - (1 - \psi_L^*)^2$ to a low firm and produce output y_L . The total output Y^* per worker in this equilibrium is therefore given by

$$Y^* = q_{HH}^* \left(1 - (1 - \psi_H^*)^2\right) + (1 - q_{HH}^*) \left(1 - (1 - \psi_L^*)^2\right) y_L. \quad (6)$$

The main question of this paper is whether the equilibrium value q_{HH}^* is constrained efficient. In order to answer this question we consider a social planner who maximizes total output in the economy. The planner cannot eliminate the coordination frictions, but he can decide to which firms the workers apply. In other words, he can control q_{HH} , q_{LL} , and q_{HL} . In section 3 we allow for free entry of vacancies and let the planner also determine θ_H and θ_L . We assume that the social planner can also decide which job a worker will take if he receives both a high and a low job offer. Suppose that he sends a fraction α of those workers to the high type firm and a fraction $1 - \alpha$ to the low type firm. Then we can derive χ_{ij}^k , $i, j, k \in \{H, L\}$, which represents the probability that playing ij results in a match with a type k firm. These probabilities

are functions of α , ψ_H , and ψ_L :

$$\chi_{HH}^H = 1 - (1 - \psi_H)^2 \quad (7)$$

$$\chi_{HL}^H = \alpha\psi_H\psi_L + \psi_H(1 - \psi_L) \quad (8)$$

$$\chi_{LL}^L = 1 - (1 - \psi_L)^2 \quad (9)$$

$$\chi_{HL}^L = (1 - \alpha)\psi_H\psi_L + \psi_L(1 - \psi_H). \quad (10)$$

The remaining probabilities, like χ_{HH}^L , are equal to zero. Using this notation, we can write the per-worker output created by the high and the low types firms as

$$Y_H = q_{HH}\chi_{HH}^H + q_{HL}\chi_{HL}^H \quad (11)$$

and

$$Y_L = (q_{LL}\chi_{LL}^L + q_{HL}\chi_{HL}^L) y_L. \quad (12)$$

This implies that the social planner wants to solve the following maximization problem:

$$\max_{q_{HH}, q_{LL}, q_{HL}, \alpha} Y = \max_{q_{HH}, q_{LL}, q_{HL}, \alpha} q_{HH}\chi_{HH}^H + q_{HL}\chi_{HL}^H + (q_{LL}\chi_{LL}^L + q_{HL}\chi_{HL}^L) y_L, \quad (13)$$

subject to $q_{HH} + q_{LL} + q_{HL} = 1$.

Solving this maximization problem gives us the optimal values q_{ij}^{**} and α^{**} , which can be used to calculate Y^{**} , the level of output. We start by showing that $\alpha^{**} = 1$, i.e. when a worker gets a job offer from both a high type and a low type firm, the planner wants him to take the high type job.

Lemma 3 *The social planner chooses $\alpha^{**} = 1$ if $q_{HL}^{**} > 0$.*

Proof. Suppose that the planner has set q_{HH}^{**} , q_{HL}^{**} , and q_{LL}^{**} to their optimal values and only still has to decide on the value of α . Note that $\frac{\partial \chi_{HH}^H}{\partial \alpha} = \frac{\partial \chi_{LL}^L}{\partial \alpha} = 0$ and that $\frac{\partial \chi_{HL}^H}{\partial \alpha} = -\frac{\partial \chi_{HL}^L}{\partial \alpha} = \psi_H\psi_L$. Therefore,

$$\left. \frac{\partial Y}{\partial \alpha} \right|_{q_{ij}^{**}} = q_{HL}^{**} \psi_H^{**} \psi_L^{**} (1 - y_L). \quad (14)$$

As long as $q_{HL}^{**} > 0$, this expression is strictly larger than zero, implying that the social planner will set α^{**} equal to its upper bound: $\alpha^{**} = 1$. When $q_{HL}^{**} = 0$, the output Y does not depend on α . In that case, the planner can choose any value. ■

The intuition for this result is clear. If a worker receives a job offer from both a high and a low firm, he must always take the job at the high type firm because his marginal productivity is higher there. Next, we can formally prove that the mixed strategy market equilibrium is inefficient: the social planner creates a higher output.

Proposition 4 *The market equilibrium defined in proposition 2 is not constrained efficient.*

Proof. First, note that proposition 2 discusses the case $y_L > \frac{\theta_H^2}{4} \left(1 - \exp\left(-\frac{2}{\theta_H}\right)\right)^2$. This implies a market equilibrium in which a strictly positive fraction of the workers sends two applications to the high sector and another strictly positive fraction sends two applications to low type firms. Next, consider a social planner who faces this equilibrium. One way in which he can increase output is by selecting a worker that plays HH and a worker that plays LL and by letting them both diversify their applications amongst the sectors. By matching HH -workers and LL -workers in this way, the total number of vacancies in each sector remains constant, implying that the matching probabilities ψ_H^* and ψ_L^* do not change. So, let the planner impose $\alpha = 1$, $q_{HH} = q_{HH}^* - \frac{1}{2}q_{HL}$, q_{HL} , and $q_{LL} = q_{LL}^* - \frac{1}{2}q_{HL} = 1 - q_{HH}^* - \frac{1}{2}q_{HL}$, where the market equilibrium corresponds to $q_{HL} = 0$. The output Y in that case equals

$$Y = \left(q_{HH}^* - \frac{1}{2}q_{HL}\right) \left(1 - (1 - \psi_H^*)^2\right) + q_{HL} (\psi_H^* + \psi_L^* (1 - \psi_H^*) y_L) + \left(q_{LL}^* - \frac{1}{2}q_{HL}\right) \left(1 - (1 - \psi_L^*)^2\right) y_L. \quad (15)$$

Taking the derivative with respect to q_{HL} gives

$$\begin{aligned} \frac{\partial Y}{\partial q_{HL}} &= \frac{1}{2} (\psi_H^{*2} - 2\psi_H^* \psi_L^* y_L + \psi_L^{*2} y_L) \\ &> \frac{1}{2} (\psi_H^* - \psi_L^* \sqrt{y_L})^2 \geq 0. \end{aligned} \quad (16)$$

This expression is strictly positive for all q_{HL} . Hence, the market equilibrium is not constrained efficient. ■

From this proof it is immediately clear that q_{HH} and q_{LL} cannot both be strictly larger than zero in the planner's solution. The planner can continue with matching HH -workers and LL -workers and thereby increasing output until one of both groups is completely exhausted. Note that although the resulting situation generates a higher social welfare than the market equilibrium, there is no reason to believe that it is the optimum. Other strategies might increase welfare even more. Unfortunately, an explicit expression for the planner's solution cannot be derived, because of the noninvertibility of ψ_i and χ_{ij}^i . Therefore, we maximize equation (13) numerically.⁷

We find that for many values of $\{\theta_H, \theta_L, y_L\}$ the planner lets all workers play HL . This is for example the case for $\theta_H = \theta_L \leq 0.5$ and $y_L \in (0, 1)$. As mentioned above, this contrasts with the decentralized market where nobody plays HL . Workers do not play HL because they are only interested in getting two job offers in the same sector. However, from the planner's point of view two job offers to the same worker is always inefficient, because in that case one firm remains unmatched, while it could have matched with a worker without any job offers. Hence, all workers ideally receive only one job offer. The planner can however not coordinate the job offers, so the only way in which he can reduce the coordination problem is by spreading the applications as much as possible, i.e. by playing HL . The planner only considers HH or LL if (i) the productivity of the L -types firms is very low, (ii) the number of firms in the market is very large, or (iii) there is a large difference between the number of high type firms and the number of low type firms.

⁷The numerical results in this paper are obtained using Ox version 3.40 (see Doornik, 2002).

Next, we consider the ratio $\frac{Y^*}{Y^{**}}$, i.e. the ratio between the total output in the decentralized equilibrium and the output level created by the social planner. This ratio is displayed in Figure 3. This figure confirms that the decentralized equilibrium is in general not efficient. The output in the mixed strategy equilibrium is only equal to the optimal level for $y_L = 1$ because then there is essentially no difference between high and low firms. For $y_L = 0$, the market equilibrium is not efficient for $\theta = \theta_H + \theta_L = \frac{1}{2}$ or 1 because the optimal number of applications per worker to the H -sector is smaller than 2 for those values of θ . The planner can use the L -sector as "garbage can" to reduce the number of applications to the H -sector which reduces the probability that two firms consider the same candidate. For $\theta = \frac{3}{2}$, the optimal number of applications is equal to 2 and the market equilibrium is constrained efficient. We also see that for low values of y_L , the equilibria with high θ perform relatively well as compared to the planner's choice, while for high values of y_L , the equilibria with low θ are closer to the constrained optimum. In the first case, almost all workers play HH , which makes the second coordination friction large (many H -firms lose their candidate to a rival firm). When θ is large, this second coordination friction is less severe. For larger values of y_L , it is less desirable to play HH because L -firm matches become more valuable but for high θ , $\frac{\partial q_{HH}}{\partial y_L}$ is smaller (see Figure 2), so q_{HH}^* adjusts too slow and therefore the low- θ equilibria are closer to the planner's solution.⁸

The social planner does not only generate a higher level of social welfare but also a lower unemployment rate than the market. This is shown in Figure 4 for $\theta_H = \theta_L = 0.5$. In the planner's solution approximately one third of the workers remains unemployed. This unemployment rate does not depend on y_L , reflecting the fact that the social planner always plays HL and $\alpha = 1$ for the chosen values of θ_H and θ_L . On the other hand, the unemployment rate in the market does depend on y_L : it decreases from 0.57 for $y_L = 0$ to 0.32 for $y_L = 1$ and is always higher than in the social planner's solution. The intuition behind this result is simple: for small values of y_L (almost) all workers in the market play HH , which causes large coordination frictions and thus a high unemployment rate. If y_L increases, a larger fraction of the workers starts to apply to low type vacancies (see Figure 2). This reduces the coordination frictions, since the same number of applications is now spread over more vacancies. As a result, the number of workers who fail to match decreases. The social planner minimizes the coordination frictions by letting everybody play HL and setting $\alpha = 1$.

Figure 5 shows the ratio between the number of matches in the high and the low sector. Again, this ratio is constant for the social planner. In the market this ratio is very high for low values of y_L , which is caused by the fact that (almost) all workers play HH in that case. The low value of y_L implies that a worker can hardly earn anything in the low sector, even if he gets two offers. Therefore, all workers try to get two offers in the high sector, even though the probability that this occurs is small. If y_L increases, the ratio between the number of matches in the high and the low sector decreases, eventually becoming equal to one for the homogenous case, i.e. $y_L = 1$.

The model has two important characteristics that could both potentially cause the inefficiency: (i) the fact

⁸Note that we do not say that the low θ equilibria are more desirable. Decreasing θ decreases output but the Planner's output decreases as well.

that workers in the decentralized market never play HL , while the social planner does and (ii) the fact that workers can not direct their applications to specific firms. Below we prove that our results are not driven by (ii). First, we show that (i) is neither solely responsible for the inefficiency: when we do not allow the social planner to let workers play HL , then he still does better than the market.

Proposition 5 *A social planner who cannot impose HL , but only HH and LL , generates a higher output than the decentralized market.*

Proof. The only decision variable for a planner who cannot impose HL is q_{HH} , the fraction of workers sending two applications to the high type sector. The remaining workers, a fraction $q_{LL} = 1 - q_{HH}$, applies twice to low type firms. Hence, output in this case equals

$$Y = q_{HH} \left(1 - (1 - \psi_H)^2\right) + (1 - q_{HH}) \left(1 - (1 - \psi_L)^2\right) y_L. \quad (17)$$

The derivative of output with respect to q_{HH} is equal to

$$\frac{\partial Y}{\partial q_{HH}} = \psi_H^2 + 2e^{-\phi_H} (1 - \psi_H) - \psi_L^2 y_L - 2e^{-\phi_L} (1 - \psi_L) y_L. \quad (18)$$

Evaluating this expression in the market equilibrium gives

$$\left. \frac{\partial Y}{\partial q_{HH}} \right|_{q_{HH}^*} = 2e^{-\phi_H^*} (1 - \psi_H^*) - 2e^{-\phi_L^*} (1 - \psi_L^*) y_L, \quad (19)$$

which typically is not equal to zero. ■

This result implies that the inefficiency does not only depend on the fact that workers fail to diversify their applications over the sectors. Part of the inefficiency arises due to the nonoptimal way in which workers choose between HH and LL . For example, for $\theta_H = \theta = 0.75$ it turns out that workers play LL too often as compared to HH for most values of y_L . However, it is good to note that, although strictly positive, the level of inefficiency is relatively small. For the considered values of θ_H and θ_L , it never exceeds 1%.

2.5 Directed search equilibrium

In this subsection, we investigate to what extent the inefficiency in our model depends on the assumption of random search. In other words, we check whether efficiency would be restored if we allow workers to direct their applications to specific firms. We find that this is not the case. General expressions for an equilibrium in a directed search framework are hard to derive, but the equilibrium outcomes of our model coincide with the equilibrium outcomes of a directed search model for many values of θ_H , θ_L , and y_L , as we state in the following proposition.

Proposition 6 *Assume that k small enough to guarantee that all workers send two applications.⁹ Then, for θ_H and θ_L sufficiently small or for y_L sufficiently large, the equilibrium outcomes described in section*

⁹ Under directed search we can have an equilibrium with $a = 1$ for some values of k . Since this is a special case of the model described in Shimer (2005), we focus on sufficiently low values of k such that $a = 2$.

2.3 are the same as in the directed search version of our model where workers observe all wages before they apply.

Proof. See appendix. ■

Figure 6 shows for which values of $\theta_H = \theta_L = \frac{1}{2}\theta$ and y_L the random search equilibrium values are the same as the directed search equilibrium values. As we prove in the appendix, only for the low type sector there exist configurations for which there is a profitable deviation from the candidate equilibrium where all firms post $w_L = 0$. For example, if there are many firms relative to workers or if the low type firms have a low productivity, which makes it unattractive for the workers to apply there. So under directed search with multiple applications and firm heterogeneity, the standard positive relation between posted wages and productivity can break down. In Postel-Vinay and Robin (2002) this happens for similar reasons. In their model, workers agree to accept a lower initial wage at high productivity firms because of future possibilities of wage increases through Bertrand competition with rival firms. In the directed search version of our model, high productivity firms always get away with posting the reservation wage while low productivity firms do not because the payoff of receiving multiple offers from high productivity firms is more attractive than from low productivity firms.

The fact that the equilibrium values under random search and directed search can coincide implies that the inefficiency of the decentralized equilibrium can not be eliminated by making search fully directed. This result is contradictory to for example Burdett, Shi and Wright (2001) and Moen (1999), who found that the equilibrium in their directed search models was constrained efficient.

To sum up, for a fixed supply of vacancies the market equilibrium is inefficient predominantly owing to workers never playing HL . Playing HL has the advantage that more H -matches can be realized by setting $\alpha = 1$ (in case of two offers, always take the H -offer). Therefore, the coordination frictions are larger than necessary. Interestingly, Galenianos and Kircher (2005) also find that worker's market portfolios of applications are socially inefficient. They only have ex ante competition for workers and show that even if workers and firms are homogeneous, workers have a desire to diversify and firms respond to this desire by offering different wages. In their model, workers choose to apply both to the high and the low wage firms but with a higher probability to the high wage firms whereas it would be socially efficient if workers apply to each firm with equal probability. Finally, note that in Albrecht et al. (2006) the portfolio inefficiency is absent because they consider both identical workers plus jobs and allow for ex post competition. They show that entry is excessive when workers apply to multiple jobs. In this section we fixed θ_i , so their inefficiency does not arise here. In section 3 we relax this assumption to see whether the entry decision is also distorted in our model.

2.6 Robustness

In this subsection we discuss to what extent our results are sensitive to the following four simplifying assumptions we made: (i) there are only two firm types, (ii) a worker cannot send more than two applications, (iii) if a firm fails to hire its candidate it cannot make an offer to the next candidate, and (iv) firms that compete for the same worker engage in Bertrand competition.

More than two firm types

Suppose there are N rankable firm types where $y_{n+1} > y_n$. Then it is straightforward to show that workers never diversify because the application-portfolio strategy, $(n+i, n)$, is dominated by (n, n) . The only way for workers to receive a positive payoff is by getting two job offers. For both portfolios, Bertrand competition leads to a wage of y_n but because the expected queue length is shorter in the least productive sector, the probability of receiving two offers is larger for the (n, n) than for the $(n+i, n)$ portfolio. One can easily generalize proposition 4 to show that also in this case the market outcome is inefficient. Therefore, considering only two firm types is not restrictive.

More than two applications

The second simplifying assumption is that a worker cannot send more than two applications. Allowing workers to apply to more than two jobs makes the analysis more difficult but does not change the nature of the portfolio problem. Still workers are only interested in the productivity-weighted probability to get more than one job offer, while the social planner wants to spread applications in order to reduce the coordination frictions. So, the fact that we restrict the workers to at most two applications is not driving our main result. If we allow workers to send three applications, (HHL) can be a symmetric equilibrium portfolio for very large θ_L and θ_H and y_L . The L -application is used to increase the probability of two offers. θ_L must be sufficiently large to make this effect large enough, y_L must be sufficiently large to make the payoffs of HL -offers close to the payoffs of HH -offers and θ_H should be sufficiently large that it is not profitable to play (HHH) . If workers apply to four jobs there exist more equilibria with diversification. Suppose $\theta_L \rightarrow \infty$, then for y_L sufficiently high, workers will send two applications to the L -sector which will result in two offers with a probability close to one. The marginal contribution of sending the remaining two applications to the L -sector are close to zero so they can best be sent to the H -sector. For five and more applications we cannot rule out regions where workers send three applications to the L -sector and the rest to the H -sector. This only happens for θ_L sufficiently large but smaller than one. The L -applications are used to secure a job while the H -applications are used to get a large payoff. We do know for sure that workers never send just one application to the H -sector $\forall a$ because the resulting wage in case of HL -offers equals the wage in case of LL -offers but the probability of occurrence is higher for the LL -portfolio.

The desire to diversify in our model is less than in Chade and Smith (2006) or Galenianos and Kircher (2005) who only have ex ante competition but no ex post competition for workers. This is caused by the fact that in our model the wage is not determined by the productivity at the most productive firm but by the

productivity of the second-highest-productivity firm that makes an offer. In the portfolio problems that they consider, the firms commit *ex ante* to a wage. Under *ex post* competition, workers have incentives to generate similar offers. Allowing workers to send more than two applications will not restore efficiency because the planner will reduce coordination frictions by letting workers diversify as much as possible between sectors while workers have strong incentives to send applications to the same sector.

Finally, note that in our setting the marginal improvement algorithm (MIA) of Chade and Smith (2006) does not work. This algorithm first picks the application with the highest expected payoff, the next application is sent to the location with the highest marginal improvement and so on and so forth. If the marginal contribution of an application is negative then the previous one is the final application. In our setting, the first application has a negative marginal payoff. Moreover, if an agent has played *LL*, an additional *H*-application always has a smaller marginal contribution to the portfolio than a single *L*-application but as we argued before, for some configurations, the *LLHH*-portfolio dominates the *LLLL*-portfolio. This makes it computationally hard to find the optimal portfolio for the case with many firm types and many applications.¹⁰

Multiple job offers

The third important assumption is that firms can offer the job to one worker only. This can be restrictive even if we assume that the marginal productivity of a second worker is zero. For example, it can be profitable for a firm to increase its matching probability by offering the same job to more applicants. The drawback of this strategy is that the firm then runs the risk that more than one worker accepts the offer. In that case, the firm has to pay a wage to all the workers it hires, while only one of them can be used in producing output.

Deriving the optimal strategy in such a model is not straightforward. First, timing matters. Suppose that a firm gives two job offers. Initially, it offers a wage equal to zero to both applicants. If one of the candidates has also received another offer, the firm must decide whether it will compete for this worker. The strategy of the firm depends on the result of the second job offer it has made. Therefore, one must make assumptions about the exact moment at which the firm learns the result of each job offer.

One way to solve the timing problem is by assuming that if their candidate has multiple offers, the firms participate in a second-price sealed bid auction, rather than Bertrand competition.¹¹ In that case all firms submit one bid w_i and the bids are revealed simultaneously. The winning firm hires the worker and pays a wage equal to the bid of the competing firm (and zero if there was no competing firm). If firms can make only one job offer, it is optimal for them to bid the productivity level, $w_i = y_i$. Hence, in that case the payoffs are identical to the payoffs described in the previous sections, i.e. under the assumption of Bertrand competition.

If firms can however make more than one job offer, deriving the optimal wage offer remains difficult. First, it is relevant whether the other offer of the firm's candidate is at a firm with multiple candidates or not. If

¹⁰There may exist algorithms where the marginal contribution of pairs or triples of applications can be used rather than comparing complete portfolios with each other but we have not been able to prove this.

¹¹See Julien et al. (2000), Kulti (1999) and Shimer (1999).

it is not, the other firm will bid more aggressively. Second, there is no pure strategy equilibrium because each candidate equilibrium wage pair is dominated by either offering one of the candidates a zero wage or offering them ε more. This is essentially the well known Burdett-Judd (1983) argument. An alternative is the shortlisting assumption of Albrecht et al. (2006) where firms pick a first candidate and a second candidate to whom they offer the job (if she is still available) in case they fail to hire their first candidate. At each of the firms they apply to, workers can be in three possible states: first candidate, second candidate or neither. This makes the algebra tedious but the bottom line is that none of the coordination frictions is eliminated. Even if a firm makes b job offers, it is still possible that it remains unmatched, because all the workers accepted offers from other firms. Moreover, workers still only care about receiving two offers while the planner wants to maximize the output-weighted number of matches. Finally, Gautier et al. (2005) and Kircher (2005) consider the case where firms can consider as many applicants as they like. Kircher shows that if firms commit to their posted wage, the directed search equilibrium is efficient. If firms can increase their initial bids, in case their (final) candidate has multiple offers, the remaining equilibrium remains inefficient.

No Bertrand Competition

The final assumption concerns Bertrand competition for workers with two offers. Alternatives are for example commitment of the firms to their initial bids, as in Galenianos and Kircher (2005) or offer-beating strategies as in Albrecht et al. (2006). Assuming commitment is basically a restriction on the firm's strategy space. Offer-beating strategies expand the firm's strategy space. Basically, the thread of Bertrand competition can reduce ex post competition and typically multiple equilibria arise. Reducing competition for workers implies that a larger part of the surplus goes to the firms and consequently entry increases. We saw that for low entry cost of H -firms, vacancy supply in both sectors was already excessive so reducing ex post competition can never generically increase efficiency. Finally, as argued before, if there only is ex ante competition for workers, only the highest offer is relevant and workers will have a stronger desire to diversify.

3 The Goods Market and Free Entry

3.1 Setting of the Game

The aim of this section is to investigate whether heterogeneity distorts entry decisions under multiple applications. Therefore, we extend the basic model by introducing free entry of firms. Before creating a job opening, firms need to buy one unit of installment capital which costs c_H for high type firms and c_L for low type firms. If a firm matches with a worker, then it can use the value of the output to cover these costs. Otherwise, it incurs a loss. Risk-neutral firms enter until the point where expected benefits are zero.

Before we continue, it is good to note that the model discussed in the previous section implicitly assumed that the output created in the low type sector and in the high type sector were perfect substitutes to each other. Free entry is not so interesting in that case because usually a corner solution is obtained where either it is more profitable to create a low type vacancy or it is more profitable to create a high type vacancy.

Therefore, we focus on a specification in which the goods are imperfect substitutes. Then we get an internal solution where both the L - and the H -commodity are produced.

Both types of firms now produce the same amount of output in case of a match ($y_H = y_L = 1$), but the value of these outputs on the goods market may differ. Those values are denoted by $p_H = 1$ (after normalization) and p_L respectively.¹² The demand on the goods market is determined by the workers who receive utility from consuming the high and the low commodity according to the following Cobb-Douglas utility function with the exogenously given constant $0.5 < \lambda < 1$:¹³

$$u(x_H, x_L) = x_H^\lambda x_L^{1-\lambda}, \quad (20)$$

where x_i represents the consumption of commodity i . Consumers maximize this utility function under the budget constraint

$$x_H + p_L x_L \leq w, \quad (21)$$

where w denotes the wage of the worker. Basically, output from both sectors is traded in a competitive goods market where λ reflects the relative preference for the H -good. Individuals have love for variety and therefore strictly positive quantities of both goods are consumed. The other characteristics of the model remain the same. Workers still send two applications and firms can increase their initial bid in case their candidate receives multiple offers.

3.2 Decentralized Market

Several of the results derived for the basic model carry over to this more extended version. For example, it remains optimal for all firms to initially post a wage equal to zero. Again, if a worker receives two job offers, the firms will increase their bids and Bertrand competition pushes the wages to the marginal product. Therefore, the expected wage of a worker who applies twice to a type i firm is equal to $\psi_i^2 p_i$, the probability of receiving two job offers multiplied by the value of the output of a type i firm.

The main difference with the model of the previous section is that workers playing HL and receiving two job offers can now be hired by either the high or the low type firm. Which firm hires depends on the value of p_L , which now is an endogenous variable. As long as $p_L < 1$, the high type firm wins the Bertrand game and hires the workers at a wage p_L . On the other hand, if $p_L > 1$ the worker matches with the low type firm at a wage equal to 1. In the case that $p_L = 1$ both firms employ the worker with probability $\frac{1}{2}$. Hence, the expected wage of a worker who plays HL is $\psi_H \psi_L \min\{1, p_L\}$.

However, again one can show that HL is dominated by either HH or LL . The proof is similar to the one in lemma 2. Only if $p_L = 1$ and $\psi_H = \psi_L$, workers are indifferent between playing HH , LL , and HL , but

¹²The assumption $y_L = 1$ is without loss of generality, since only the total value of the output, i.e. $y_L p_L$, is relevant in our analysis. Fixing y_L to a value different from 1 therefore only implies a rescaling of p_L .

¹³Note that the labels *high* and *low* no longer refer to the productivity of a firm, since the productivity is assumed to be the same for both types. We nevertheless stick to these labels in order to keep notation consistent. Instead, one can interpret the labels in the following way: high type firms create a commodity that has a larger weight ($\lambda > \frac{1}{2}$) in (20) than the commodity created by the low type firms ($1 - \lambda < \frac{1}{2}$).

this is only because in that case all jobs are identical. In all other cases, workers will only consider playing HH and LL .

A firm of type i has a positive revenue if it attracts at least one applicant and if the worker to which it offers the job, does not receive a second job offer. The first event happens with probability $(1 - e^{-\phi_i})$, while the probability of the latter equals $(1 - \psi_i)$.¹⁴ Therefore, the expected profit of such a firm equals

$$\pi_i = (1 - e^{-\phi_i}) (1 - \psi_i) p_i - c_i, \quad (22)$$

which under free entry is equal to zero in equilibrium. From this, one can see that an equilibrium in which HL is not strictly dominated, i.e. with $p_L = 1$ and $\psi_H = \psi_L$, can only arise if $c_H = c_L$.

In equilibrium, the ratio of the prices of the commodities must equal the (absolute value of the) marginal rate of substitution (MRS):

$$\frac{p_L}{p_H} = \frac{\partial U / \partial x_L}{\partial U / \partial x_H} \Big|_{x_H=Y_H, x_L=Y_L} = \frac{1 - \lambda Y_H}{\lambda Y_L}. \quad (23)$$

The expected per-worker output created by the high type firms is $q_{HH} (1 - (1 - \psi_H)^2)$, while the low type firms produce $(1 - q_{HH}) (1 - (1 - \psi_L)^2)$ per worker. So, equation (23) is equivalent to

$$p_L = \frac{1 - \lambda}{\lambda} \frac{q_{HH}}{1 - q_{HH}} \frac{1 - (1 - \psi_H)^2}{1 - (1 - \psi_L)^2}. \quad (24)$$

Summarizing we can define the equilibrium as follows:

Definition 1 *An equilibrium in the decentralized market is a tuple $\{p_L, \theta_H, \theta_L, q_{HH}\}$ such that the following four conditions hold:*

$$\psi_H^2 = \psi_L^2 p_L \quad (25)$$

$$p_L = \frac{1 - \lambda}{\lambda} \frac{q_{HH}}{1 - q_{HH}} \frac{1 - (1 - \psi_H)^2}{1 - (1 - \psi_L)^2} \quad (26)$$

$$(1 - e^{-\phi_H}) (1 - \psi_H) = c_H \quad (27)$$

$$(1 - e^{-\phi_L}) (1 - \psi_L) = \frac{c_L}{p_L} \quad (28)$$

Equation (25) represents the indifference condition for the workers, while equation (26) makes sure that the price of the low commodity equals the MRS. Equation (27) and (28) are the zero-profit conditions for the high and low type firms respectively. Next, we can show that there is a unique equilibrium.

Proposition 7 *In a decentralized market a unique equilibrium $\{p_L^*, \theta_H^*, \theta_L^*, q_{HH}^*\}$ exists $\forall 0 < c_H, c_L < 1$.*

Proof. See appendix. ■

This extended version of the model also has three exogenous parameters, c_H , c_L , and λ . The following proposition summarizes how the equilibrium is affected by a change in c_L , the entry cost of the low type firms.

¹⁴Due to the infinite size of the labor market, these events are independent.

Proposition 8 ϕ_L^* , p_L^* , q_{HH}^* , and θ_H^* are strictly increasing in c_L , while θ_L^* is strictly decreasing in c_L . Finally, ϕ_H^* is not affected by a change in c_L .

Proof. See appendix. ■

The intuition behind proposition 8 is the following. An increase in c_L reduces profits for low type firms and therefore fewer L -vacancies are opened. This makes it relatively more attractive to apply to the high type firms, implying that q_{HH}^* increases. This pushes up the profits for high type firms, which induces more high type vacancies to be opened. This increase in θ_H^* exactly offsets the increase in q_{HH}^* such that the probability to get a job after applying to a high type firm remains constant.

The above proposition also allows us to compare the high type vacancies with the low type vacancies. It turns out that the entry cost is decisive for which type of vacancy receives more applications and receives a higher price for the created output:

Corollary 2 *In the decentralized equilibrium, the vacancy type with the higher entry cost receives more applications, provides the worker with a smaller probability of getting a job offer, and has a higher price for the associated produced commodity.*

Proof. The proof is similar to the proof of the previous proposition with \hat{p}_L^* replaced by $p_H^* \equiv 1$, $\hat{\phi}_L^*$ by ϕ_H^* , $\hat{\psi}_L^*$ by ψ_H^* , and \hat{c}_L by c_H . ■

The reason that λ has no effect on the expected queue length in a sector is that an increase in λ decreases p_L but increases θ_H/θ_L . More applications will go to the H -sector but there will also be more vacancies in the H -sector. Both effects offset each other.

The second exogenously given parameter is the entry cost for the high type firms. An increase in this parameter decreases the price of the low commodity, but increases the expected number of applications to both high and low type vacancies. This is summarized in the following proposition:

Proposition 9 ϕ_H^* and ϕ_L^* are strictly increasing in c_H , while p_L^* is strictly decreasing in c_H .

Proof. The free entry condition for the H -firms shows that an increase in c_H strictly increases ϕ_H^* . This means that ψ_H^* strictly decreases. For the indifference condition to continue to hold, $\psi_L(\phi_L^*(p_L^*))^2 p_L^*$ has to decrease as well. Since $\psi_L(\phi_L^*(p_L))^2 p_L$ is strictly increasing in p_L (see proof of proposition 7), a decrease in ψ_H^* implies a decrease in p_L^* and therefore an increase in ϕ_L^* . ■

It is not trivial to analytically derive the effect of an increase in c_H on q_{HH}^* , θ_H^* and θ_L^* . Since we only have three exogenous parameters we can rely on numerical computations. We find that the above three variables are decreasing in c_H . Figures 7, 8 and 9 respectively show q_{HH}^* , θ_H^* and θ_L^* as a function of c_H for several values of c_L , where $\lambda = 0.6$.

The last exogenously given parameter is the preference parameter λ , i.e. the share of income that the workers spend on consuming the high commodity. Not surprisingly, an increase in λ turns out to have a positive effect on the fraction of workers applying to the high vacancies and on the number of high vacancies. However, it causes a decrease in the number of low vacancies. The change in the number of vacancies exactly offsets the change in q_{HH}^* , so that the expected number of applications per vacancy does not change..

Proposition 10 q_{HH}^*, θ_H^* are strictly increasing in λ , while θ_L^* is strictly decreasing in λ . A change in λ does not affect ϕ_H^*, ϕ_L^* , and p_L^* .

Proof. ϕ_H^* is determined by equation (27) only and therefore not affected by a change in λ . This means that the left hand side of equation (25) remains constant. As we showed in proposition 7, the right hand side of this equation is strictly increasing in p_L , implying that p_L^*, ψ_L^* , and ϕ_L^* do not change either. From the fact that p_L remains constant and λ increases, we can derive that in equation (26) the factor $\frac{q_{HH}}{1-q_{HH}}$ must increase. Since the first derivative of this expression is strictly positive, this means that q_{HH} has to increase. Now it is straightforward to show that θ_H^* must increase and θ_L^* must decrease in order to keep ϕ_H^* and ϕ_L^* fixed. ■

3.3 Efficiency

Since we allow for free entry, we can now test whether the number and composition of vacancies is constrained efficient. Specifically, we assume that the social planner can again determine q_{HH}, q_{LL}, q_{HL} , and α , like in the basic model, but now he can also determine the number and composition of firms in the market, θ_H and θ_L . Using the same definitions for χ_{ij}^k as in section 2.4, we can write Y_i , i.e. total output created by type i firms, as follows:

$$Y_H = q_{HH}\chi_{HH}^H + q_{HL}\chi_{HL}^H \quad (29)$$

and

$$Y_L = q_{LL}\chi_{LL}^L + q_{HL}\chi_{HL}^L. \quad (30)$$

Next, denote the net value of the output per worker by V :

$$V = p_H Y_H + p_L Y_L - \theta_H c_H - \theta_L c_L.$$

The social planner is not concerned with redistribution issues. He just wants to maximize social welfare, i.e. the utility that can be obtained from V . This implies that he maximizes the indirect utility function associated to the Cobb-Douglas utility function specified in equation (20):

$$\max_{q_{HH}, q_{LL}, q_{HL}, \alpha, \theta_H, \theta_L} \left(\frac{\lambda V}{p_H} \right)^\lambda \left(\frac{(1-\lambda)V}{p_L} \right)^{1-\lambda} \quad (31)$$

under the condition that $q_{HH} + q_{LL} + q_{HL} = 1$. Again, the price of the low commodity has to be equal to the marginal rate of substitution. Therefore, we can rewrite equation (31) as follows:

$$\max_{q_{HH}, q_{LL}, q_{HL}, \alpha, \theta_H, \theta_L} (Y_H - \lambda \theta_H c_H - \lambda \theta_L c_L) \left(\frac{Y_L}{Y_H} \right)^{1-\lambda}. \quad (32)$$

The corresponding system of first order conditions cannot be solved analytically. Therefore, we use numerical optimization methods to derive the optimal values q_{HH}^{**} , q_{LL}^{**} , q_{HL}^{**} , α^{**} , θ_H^{**} , and θ_L^{**} . The results indicate that the optimal value for q_{LL}^{**} equals 0, i.e. the social planner does not let workers play LL .¹⁵ The optimal values for $q_{HH}^{**} = 1 - q_{HL}^{**}$ are displayed in Figure 10 for several values of c_L and $\lambda = 0.6$.¹⁶ Each line shows two clear jumps. The first jump occurs where $c_H = c_L$, which can be explained by the behavior of α^{**} . This value is always equal to zero for $c_H < c_L$ and equal to one for $c_H > c_L$, because when a worker receives both a high and a low type offer, the planner wants the worker to fill the position that is more expensive to create. Ceteris paribus, this jump in α^{**} at $c_H = c_L$ increases the probability for a high firm to match and decreases the probability for a low firm to match. Since the output the planner wants to create in the high and the low sector does however not change discontinuously, the positive jump in α^{**} must be neutralized by a negative jump in q_{HH}^{**} .

The second jump has no clear economic meaning. It is the result of the fact that the social welfare function is non-monotonic in its parameters. The value of c_H for which this second jump occurs is negatively related to λ . For large values of λ and c_L it can happen that this jump occurs before the point where $c_H = c_L$. In that case, there is only one jump.

Next, we turn to the important question whether there are too many or too few vacancies created in the decentralized market equilibrium. Albrecht et al. (2006) prove that in their model the market always opens more vacancies than the social planner if the number of applications is fixed, but that there can be either too many or too few vacancies if the number of applications is endogenous. In our model we focus on $a = 2$, but the composition of these applications over the sectors is endogenous both for the market and the planner. Unlike in Albrecht et al. (2006), the expected number of applications that a worker sends to a specific sector can now be a non-integer value, because he can play mixed strategies with respect to the sectors he applies to. Hence, the heterogeneity amongst the firms gives both the market and the social planner more freedom in choosing the optimal number of applications, even if the total number of applications is fixed.

Figures 11 and 12 respectively display the number of H -vacancies and the number of L -vacancies created by the market and the planner as a function of the entry cost for type H firms. The entry cost for type L firms is fixed at 0.5, while λ is still assumed to be 0.6.¹⁷ The Figures show that either too many or too few vacancies (both high and low) are opened in the decentralized market, depending on the values of the exogenous parameters. For low values of c_H , the market opens too many vacancies (both high and low) compared to the social optimum. The intuition is that because the posted wages are driven to zero, firms basically have monopsony power. The existence of ex post competition only partly offsets this. Albrecht et al. (2006) show for the identical-workers-and-jobs case that efficiency requires full ex ante and ex post competition.¹⁸ When c_H approaches 1, the Albrecht et al. (2006) case is reversed, the social planner creates

¹⁵This conclusion even holds for c_L close to 1 and λ close to 0.5.

¹⁶Fixing λ at a different value, e.g. 0.9, changes the values of q_{HH}^{**} , q_{LL}^{**} , q_{HL}^{**} , α^{**} , θ_H^{**} , and θ_L^{**} , but none of the qualitative conclusions in this section.

¹⁷Different values of c_L and λ do not affect the main conclusions.

¹⁸In their directed search model, the possibility of ex post competition eliminates the ex ante competition for workers.

more vacancies than the market.

The intuition for the latter result is the following. If c_H approaches 1 in the decentralized market, no high firm is willing to enter, because its expected payoff is negative in that case. However, without supply of the high type commodity, workers can never obtain a positive utility and therefore the entire market collapses: there are no firms active in equilibrium. This result depends on our assumption that the cross partials of the utility function are positive. The H -firms do not internalize that increasing their output increases the value of L -output in particular when Y_H is low.

In order to check whether the market is constrained efficient, we compare the ratio between the utility obtained in the decentralized equilibrium (i.e. the indirect utility function evaluated at the equilibrium values) and the utility associated with the social planner's solution. This ratio is displayed by the dashed line in Figure 13 for $\lambda = 0.6$. It shows that for small c_H , market utility is about 80% of what could be achieved. As c_H increases, the inefficiency goes up and when c_H approaches 1, the ratio of the utility obtained in the market and under the planner goes to zero.

The intuition for this result is the same as above: in the decentralized market no vacancies are created if c_H approaches 1. The social planner however does create vacancies in that case. So, for c_H close enough to 1, the created output is virtually zero in the decentralized market but strictly positive under the social planner. This implies that the relative efficiency of the decentralized market equilibrium goes to 0.

To see to what extent this inefficiency is caused by the fact that the planner plays HL , we also consider a constrained planner who can only play (a mixture of) HH and LL . The efficiency of the decentralized equilibrium relative to this constrained planner's optimum is displayed in Figure 13 by the solid line. The line shows that in this case the inefficiency is almost as large as in the case with the unconstrained planner. This is dramatically different from the model in section 2 where most of the market inefficiency was due to the fact that workers do not play HL .

The conclusions drawn in section 2.4 about employment in the high and the low sector do also not fully carry over to the extended version of the model. As can be seen in Figure 14, the unemployment rate in the market is now not always higher than under the social planner. For small values of c_H the reverse holds, which directly follows from the fact that the market opens more vacancies than the planner. Figure 15 shows that, except for extremely low values of c_H (below 0.02), the planner always generates a higher matching rate in the high sector than the market and the difference increases with c_H . This follows from the facts that (i) for low c_H , the market creates both too many H and L vacancies but the excessive number of low vacancies is larger and (ii) for higher c_H , the market creates too few H vacancies, which again is caused by the positive cross partials in (20).

To sum up, the market creates too many vacancies if the high-type-vacancy creation costs are low, while it creates too few vacancies if the high type vacancy creation costs are high. As a result of this, the expected number of applications that a high type vacancy receives in a decentralized market is larger than socially

optimal for high values of c_H and smaller than optimal for low values of c_H . A similar pattern is found for the expected number of applications received by low type firms. If we restrict the planner to only play HH and LL , this conclusion still holds. Allowing for free entry almost completely eliminates the inefficiencies caused by the fact that workers do not play HL and $\alpha = 1$, but introduces new distortions: (i) since posted wages are 0, firms have monopsony power, (ii) H firms do not internalize that more output in the H -sector increases the value of output in the L sector, see (23). (i) dominates for low vacancy cost and (ii) dominates for high vacancy cost.

4 Final Remarks

We presented a simple model where workers could apply to multiple, heterogeneous jobs. Workers do not apply to firms with the highest expected payoffs for an individual application but rather maximize the value of their portfolio. We also extend the model with free entry.

The resulting equilibrium is not efficient for two reasons. Workers want to maximize the productivity-weighted probability to get two job offers, while the planner aims to maximize the productivity-weighted number of matches. This conflict of interest results in too little matches and excessive unemployment. We showed that this result is not driven by the fact that search is random in our model. For a large share of parameter values the posted wages are also zero in the directed search version of our model as in Albrecht et al. (2006).

If we allow for free entry there is a second source of inefficiency. For high creation cost in the high productivity sector, the market creates too little vacancies. If entry cost are high, the risk of Bertrand competition makes firms stop entering the market at a point where the marginal social benefits are still positive. On the other hand if entry cost are low, vacancy creation is excessive because the absence of ex ante competition gives firms too much rents. The vacancy creation distortions can in principle be neutralized by an appropriately chosen firm tax or subsidy scheme. The workers' portfolio distortions are more severe. Governments may have instruments to make one of the sectors more attractive but this will only increase the fraction of workers who send both applications to this sector without increasing the fraction of workers that mixes between sectors.

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Appendix

A Proofs

A.1 Proof of proposition 3

Proof. (i) First, note that

$$\frac{\partial \psi_i}{\partial \phi_i} = \frac{1}{\phi_i^2} ((1 + \phi_i) e^{-\phi_i} - 1) < 0, \quad (33)$$

since $e^{\phi_i} > 1 + \phi_i \forall \phi_i > 0$. Using this and $\frac{\partial \phi_i}{\partial \theta_i} < 0$, we can derive that $\frac{\partial \psi_H^2}{\partial \theta_H} = 2\psi_H \frac{\partial \psi_H}{\partial \phi_H} \frac{\partial \phi_H}{\partial \theta_H} > 0$, which means that an increase in θ_H shifts the ψ_H^2 -curve in Figure 1 upwards. On the other hand, $\frac{\partial \psi_L^2 y_L}{\partial \theta_H} = 0$, so an increase in θ_H does not affect the $\psi_L^2 y_L$ -curve. Hence, the intersection point of the two curves moves to the northeast, implying an increase in the equilibrium value q_{HH}^* and in the expected payoffs. This means that both ψ_H^* and ψ_L^* increase and both ϕ_H^* and ϕ_L^* decrease.

(ii) Note that $\frac{\partial \psi_L^2 y_L}{\partial \theta_L} = 2\psi_L y_L \frac{\partial \psi_L}{\partial \phi_L} \frac{\partial \phi_L}{\partial \theta_L} > 0$. Hence, an increase in θ_L shifts $\psi_L^2 y_L$ -curve upwards, but does not affect the ψ_H^2 -curve. Therefore, q_{HH}^* decreases, while ψ_H^* and ψ_L^* increase and, consequently, ϕ_H^* and ϕ_L^* decrease.

(iii) Finally, an increase in y_L shifts the $\psi_L^2 y_L$ -curve upwards, but does not affect the ψ_H^2 -curve. Therefore, q_{HH}^* decreases, while ψ_H^* increases. The latter implies a decrease in ϕ_H^* . Since θ_L remains constant, the decrease in q_{HH}^* results in an increase in ϕ_L^* and consequently a decrease in ψ_L^* . ■

A.2 Proof of proposition 6

Proof. Suppose that all firms posting a wage equal to zero is not a directed search equilibrium. Then a profitable deviation must exist for either the high type firms or the low types firms. Consider a deviation by a high type firm first. Instead of 0 it posts a strictly positive wage: $w'_H > 0$. Workers now have two additional application strategies: they can send (i) one application to the deviant and the other one to a high firm or (ii) one application to the deviant and the other one to a low firm. Denote the former strategy by $H'H$ and the latter by $H'L$. The payoff of playing $H'H$ equals

$$\psi'_H \psi_H + \psi'_H (1 - \psi_H) w'_H \quad (34)$$

and the payoff of $H'L$ equals

$$\psi'_H \psi_L y_L + \psi'_H (1 - \psi_L) w'_H, \quad (35)$$

where ψ'_H is defined in the usual way and denotes the probability that an application to the deviant results in a job offer.

Since we consider a large labor market, a specific worker applies with probability zero to the deviant. So, the presence of a deviant does not affect the average number of applications received by the other non-deviant high or low firms. Therefore, the indifference condition $\psi_H^2 = \psi_L^2 y_L$ must still hold. By substituting

$\psi_H = \psi_L \sqrt{y_L}$ in equation (34) and using the fact that $1 > \sqrt{y_L} > y_L$, one can easily see that $H'L$ is dominated by $H'H$.

In response to the deviation by one of the high firms, workers will adjust their application strategies such that they are indifferent between HH , LL and $H'H$. The new equilibrium is therefore defined by the following two equations:

$$\begin{aligned}\psi_H^2 &= \psi_L^2 y_L \\ \psi_H^2 &= \psi'_H \psi_H + \psi'_H (1 - \psi_H) w'_H\end{aligned}$$

Let ϕ'_H denote the expected number of applications that the deviant receives. Then, by substituting $\psi'_H = \frac{1}{\phi'_H} (1 - e^{-\phi'_H})$ in the second condition and rearranging the result, we can derive the following relation between the posted wage w'_H and ϕ'_H :

$$w'_H = \frac{1}{1 - \psi_H} \left(\frac{\phi'_H \psi_H^2}{1 - e^{-\phi'_H}} - \psi_H \right). \quad (36)$$

The first derivative of this function with respect to ϕ'_H equals

$$\frac{\partial w'_H}{\partial \phi'_H} = \frac{\psi_H^2}{\psi_H - 1} \frac{e^{-\phi'_H} + \phi'_H e^{-\phi'_H} - 1}{e^{-2\phi'_H} - 2e^{-\phi'_H} + 1} > 0 \quad \forall \phi'_H > 0.$$

Hence, w'_H is a monotonic function of ϕ'_H : the higher the wage set by the deviant, the higher the expected number of applications it receives. The fact that w'_H is monotonically increasing in ϕ'_H also implies that rather than deriving the optimal wage for a deviant, we can derive the optimal queue length. The one implies the other.

After substituting equation (36), the profit function for a high type deviant equals

$$\begin{aligned}\pi'_H &= (1 - e^{-\phi'_H}) (1 - \psi_H) (1 - w'_H) \\ &= (1 - e^{-\phi'_H}) (1 - \psi_H) \left(1 - \frac{1}{1 - \psi_H} \left(\frac{\phi'_H \psi_H^2}{1 - e^{-\phi'_H}} - \psi_H \right) \right).\end{aligned}$$

Differentiating this profit function with respect to ϕ'_H yields the following expression:

$$\frac{\partial \pi'_H}{\partial \phi'_H} = e^{-\phi'_H} - \psi_H^2,$$

which is a strictly decreasing function of ϕ'_H that equals zero for $\phi'_H = -2 \log(\psi_H)$. Therefore, the profit function has a global maximum in this point. The corresponding value of w'_H follows from evaluating equation (36) in this maximum:

$$w'_H = \frac{\psi_H (\psi_H^2 - 2\psi_H \log(\psi_H) - 1)}{(1 - \psi_H)^2 (1 + \psi_H)}. \quad (37)$$

This expression has the same sign as $\psi_H^2 - 2\psi_H \log(\psi_H) - 1$. The first derivative of this equation is equal to $2(\psi_H - \log \psi_H - 1)$, which easily can be shown to be positive for all ψ_H in the interval $(0, 1)$. Together with the fact that $\lim_{\psi_H \rightarrow 1} \psi_H^2 - 2\psi_H \log(\psi_H) - 1 = 0$, this implies that the right hand side of equation (37) is negative $\forall \psi_H \in (0, 1)$. Since we do not allow for negative wages, this optimal value of w'_H is not feasible.

Given that the profit is strictly decreasing in $\phi'_H > -2 \log(\psi_H)$ and that w'_H is strictly increasing in ϕ'_H , the profit function maximization problem therefore has a boundary solution: the deviant maximizes its profit by posting $w'_H = 0$. This implies that the best response for a potential deviant is to also post w_H .

Now we perform the same analysis for a low type deviant. Suppose that it posts a wage $w'_L > 0$. In that case the payoff of playing LL' equals

$$\psi_L \psi'_L y_L + \psi'_L (1 - \psi_L) w'_L = \psi'_L w'_L + \psi'_L \psi_L (y_L - w'_L)$$

and the payoff of HL' equals

$$\psi_H \psi'_L y_L + \psi'_L (1 - \psi_H) w'_L = \psi'_L w'_L + \psi'_L \psi_H (y_L - w'_L),$$

where ψ'_L denotes the probability that an application to the deviant results in a job offer.

In a similar way as we described above, one can show that the strategy HL' is dominated by LL' . The new equilibrium is therefore defined by the following two indifference conditions:

$$\begin{aligned} \psi_H^2 &= \psi_L^2 y_L \\ \psi_L^2 y_L &= \psi_L \psi'_L y_L + \psi'_L (1 - \psi_L) w'_L \end{aligned}$$

Let ϕ'_L denote the expected number of applications that the deviant receives. Then, by substituting $\psi'_L = \frac{1}{\phi'_L} (1 - e^{-\phi'_L})$ in the second condition and rearranging the result, we can derive the following relation between the posted wage w'_L and ϕ'_L :

$$w'_L = \frac{1}{1 - \psi_L} \left(\frac{\phi'_L \psi_L^2 y_L}{1 - e^{-\phi'_L}} - \psi_L y_L \right). \quad (38)$$

The first derivative of this function with respect to ϕ'_L equals

$$\frac{\partial w'_L}{\partial \phi'_L} = \frac{\psi_L^2 y_L}{\psi_L - 1} \frac{e^{-\phi'_L} + \phi'_L e^{-\phi'_L} - 1}{e^{-2\phi'_L} - 2e^{-\phi'_L} + 1} > 0 \quad \forall \phi'_L > 0.$$

Hence w'_L is a monotonic function of ϕ'_L : the higher the wage set by the deviant, the higher the expected number of applications it receives.

The profit function for the deviant equals

$$\begin{aligned} \pi'_L &= (1 - e^{-\phi'_L}) (1 - \psi_L) (1 - w'_L) \\ &= (1 - e^{-\phi'_L}) (1 - \psi_L) \left(1 - \frac{1}{1 - \psi_L} \left(\frac{\phi'_L \psi_L^2 y_L}{1 - e^{-\phi'_L}} - \psi_L y_L \right) \right). \end{aligned}$$

Differentiating this this profit function with respect to ϕ'_L yields the following expression:

$$\frac{\partial \pi'_L}{\partial \phi'_L} = e^{-\phi'_L} (1 - (1 - y_L) \psi_L) - \psi_L^2 y_L,$$

which is a strictly decreasing function of ϕ'_L that equals zero for $\phi'_L = -\log \kappa$, where $\kappa \equiv \frac{\psi_L^2 y_L}{1 - (1 - y_L) \psi_L}$. Therefore the profit function has a global maximum in this point. The corresponding value of w'_L follows

from evaluating equation (38) in this maximum:

$$w'_L = \frac{-\psi_L y_L}{1 - \psi_L} \left(\frac{\psi_L \log \kappa}{1 - \kappa} + 1 \right).$$

One can check that $\lim_{\psi_L \rightarrow 0} w'_L = 0$, $\lim_{\psi_L \rightarrow 0} \frac{\partial w'_L}{\partial \psi_L} = -y_L < 0$ and, by applying l'Hospital's Rule twice, $\lim_{\psi_L \rightarrow 1} w'_L = \frac{1-y_L}{2} > 0$ (see Figure 16). Therefore, it depends on the equilibrium value ψ_L^* whether a profitable deviation exists. For ψ_L^* close to 0 the optimal value for w'_L is negative. Given the fact that $\frac{\partial \pi'_L}{\partial \phi'_L} < 0$ for $\phi'_L > -\log \kappa$ and that $\frac{\partial w'_L}{\partial \phi'_L} > 0 \forall \phi'_L > 0$, this implies that low type firms have no incentive to post a wage that is different from 0. On the other hand, for ψ_L^* close to 1, it is profitable for a low firm to deviate by posting a wage that is strictly positive. It straightforward to show that both cases can occur. For example, $\psi_L^* \rightarrow 0$ if $\theta_H \rightarrow 0$, $\theta_L \rightarrow 0$ and $y_L \rightarrow 1$, while $\psi_L^* \rightarrow 1$ if $\theta_H \rightarrow \hat{\theta}_H$ where $\hat{\theta}_H$ is such that $\frac{\hat{\theta}_H^2}{4} \left(1 - \exp\left(-\frac{2}{\hat{\theta}_H}\right) \right)^2 = y_L$. ■

A.3 Proof of proposition 7

Proof. The function $1 - e^{-\phi_H}$ is strictly positive and strictly increasing $\forall \phi_H > 0$. The same is true for the function $1 - \psi_H = 1 - \frac{1}{\phi_H} (1 - e^{-\phi_H})$. Therefore, the revenue for the high firm $(1 - e^{-\phi_H}) (1 - \psi_H)$ is a strictly increasing function of ϕ_H with

$$\lim_{\phi_H \rightarrow 0} (1 - e^{-\phi_H}) (1 - \psi_H) = 0$$

and

$$\lim_{\phi_H \rightarrow \infty} (1 - e^{-\phi_H}) (1 - \psi_H) = 1.$$

This implies that the condition (27) uniquely identifies a value $\phi_H^* > 0$ for any $0 < c_H < 1$.

Since $(1 - e^{-\phi_L}) (1 - \psi_L) < 1$, a necessary condition for condition (28) to hold is that $p_L > c_L$. Assume for the moment that p_L is exogenously given such that this condition is satisfied. In that case any value $0 < c_L < 1$ uniquely identifies a value ϕ_L^* as a function of p_L , i.e. $\phi_L^*(p_L)$. Since $(1 - e^{-\phi_L}) (1 - \psi_L)$ is strictly increasing in ϕ_L , $\phi_L^*(p_L)$ is strictly decreasing in p_L with $\lim_{p_L \rightarrow c_L} \phi_L^*(p_L) = \infty$ and $\lim_{p_L \rightarrow \infty} \phi_L^*(p_L) = 0$. Using this, it follows directly that $\psi_L(\phi_L^*(p_L))$ and $\psi_L(\phi_L^*(p_L))^2 p_L$ are both strictly increasing in p_L and that

$$\lim_{p_L \rightarrow c_L} \psi_L(\phi_L^*(p_L))^2 p_L = 0$$

and

$$\lim_{p_L \rightarrow \infty} \psi_L(\phi_L^*(p_L))^2 p_L = \infty.$$

This implies that given ϕ_H^* and $\phi_L^*(p_L)$ there exists a unique value $p_L^* > c_L$ such that the indifference condition is satisfied.

Let $\phi_L^* = \phi_L^*(p_L^*)$, $\psi_H^* = \psi_H(\phi_H^*)$ and $\psi_L^* = \psi_L(\phi_L^*(p_L^*))$. Then

$$\lim_{q_{HH} \rightarrow 0} \frac{1 - \lambda}{\lambda} \frac{q_{HH}}{1 - q_{HH}} \frac{1 - (1 - \psi_H^*)^2}{1 - (1 - \psi_L^*)^2} = 0,$$

while

$$\lim_{q_{HH} \rightarrow 1} \frac{1-\lambda}{\lambda} \frac{q_{HH}}{1-q_{HH}} \frac{1-(1-\psi_H^*)^2}{1-(1-\psi_L^*)^2} = \infty,$$

and

$$\frac{d}{dq_{HH}} \frac{q_{HH}}{1-q_{HH}} = \frac{1}{(1-q_{HH})^2} > 0.$$

The Intermediate Value Theorem implies that there exists a unique value $0 < q_{HH}^* < 1$ such that p_L^* equals the MRS $\frac{1-\lambda}{\lambda} \frac{q_{HH}^*}{1-q_{HH}^*} \frac{1-(1-\psi_H^*)^2}{1-(1-\psi_L^*)^2}$. Using q_{HH}^* , ϕ_H^* and ϕ_L^* , it is straightforward to determine θ_H^* and θ_L^* . Now, the equilibrium is defined by p_L^* , θ_H^* , θ_L^* , and q_{HH}^* . ■

A.4 Proof of proposition 8

Proof. Let p_L^* and \hat{p}_L^* be equilibrium prices with $p_L^* > \hat{p}_L^*$. Using the indifference condition this implies $\psi_L^* < \hat{\psi}_L^*$, which by the definition of ψ_i is equivalent to $\phi_L^* > \hat{\phi}_L^*$. Using the inequalities $p_L^* > \hat{p}_L^*$ and $\phi_L^* > \hat{\phi}_L^*$, and the fact that $(1 - e^{-\phi_L}) \left(1 - \frac{1}{\phi_L} (1 - e^{-\phi_L})\right)$ is strictly increasing in ϕ_L , one can derive that

$$\begin{aligned} \hat{c}_L &= (1 - e^{-\hat{\phi}_L^*}) \left(1 - \frac{1}{\hat{\phi}_L^*} (1 - e^{-\hat{\phi}_L^*})\right) \hat{p}_L^* \\ &< (1 - e^{-\phi_L^*}) \left(1 - \frac{1}{\phi_L^*} (1 - e^{-\phi_L^*})\right) p_L^* = c_L \end{aligned}$$

Likewise, one can show that $p_L^* < \hat{p}_L^*$ implies $\psi_L^* > \hat{\psi}_L^*$, $\phi_L^* < \hat{\phi}_L^*$, and $\hat{c}_L > c_L$ and that $p_L^* = \hat{p}_L^*$ implies $\psi_L^* = \hat{\psi}_L^*$, $\phi_L^* = \hat{\phi}_L^*$, and $\hat{c}_L = c_L$. Since we have listed all possibilities, we can invert this result and state that $c_L > \hat{c}_L$ implies $\phi_L^* > \hat{\phi}_L^*$, $\psi_L^* < \hat{\psi}_L^*$, and $p_L^* > \hat{p}_L^*$. Hence, ϕ_L^* and p_L^* are strictly increasing in c_L . The equilibrium value of ϕ_H^* is determined by the condition (27) only and therefore not affected by a change in c_L .

Substituting the indifference condition in the fourth condition and solving for q_{HH}^* yields the following expression

$$q_{HH}^* = \frac{2\lambda\psi_H^*}{2\lambda(\psi_H^* - \psi_L^*) + (2 - \psi_H^*)\psi_L^*}.$$

>From this we can derive

$$\begin{aligned} \frac{dq_{HH}^*}{dc_L} &= \frac{\partial q_{HH}^*}{\partial \psi_L^*} \frac{\partial \psi_L^*}{\partial c_L} + \frac{\partial q_{HH}^*}{\partial \psi_H^*} \frac{d\psi_H^*}{dc_L} \\ &= \frac{\partial q_{HH}^*}{\partial \psi_L^*} \frac{\partial \psi_L^*}{\partial c_L} > 0. \end{aligned}$$

Now we have

$$\frac{\partial \phi_H^*}{\partial c_L} = \frac{\partial \phi_H^*}{\partial q_{HH}^*} \frac{\partial q_{HH}^*}{\partial c_L} + \frac{\partial \phi_H^*}{\partial \theta_H^*} \frac{\partial \theta_H^*}{\partial c_L}.$$

As shown above, $\frac{\partial \phi_H^*}{\partial c_L} = 0$, while $\frac{\partial \phi_H^*}{\partial q_{HH}^*} > 0$, $\frac{\partial q_{HH}^*}{\partial c_L} > 0$ and $\frac{\partial \phi_H^*}{\partial \theta_H^*} < 0$. This implies $\frac{\partial \theta_H^*}{\partial c_L} > 0$.

Likewise, we have

$$\frac{\partial \phi_L^*}{\partial c_L} = \frac{\partial \phi_L^*}{\partial q_{HH}^*} \frac{\partial q_{HH}^*}{\partial c_L} + \frac{\partial \phi_L^*}{\partial \theta_L^*} \frac{\partial \theta_L^*}{\partial c_L},$$

where $\frac{\partial \phi_L^*}{\partial c_L} > 0$, $\frac{\partial \phi_L^*}{\partial q_{HH}^*} < 0$, $\frac{\partial q_{HH}^*}{\partial c_L} > 0$ and $\frac{\partial \phi_L^*}{\partial \theta_L^*} < 0$. This implies $\frac{\partial \theta_L^*}{\partial c_L} < 0$. ■

B Figures

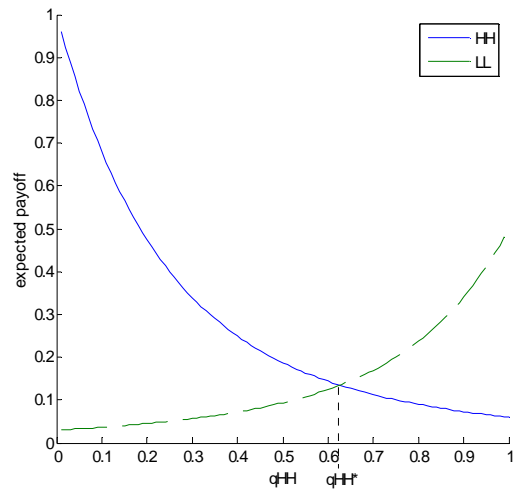


Figure 1: Expected payoff of playing HH and LL for $\theta_H = \theta_L = 1$ and $y_L = 0.5$

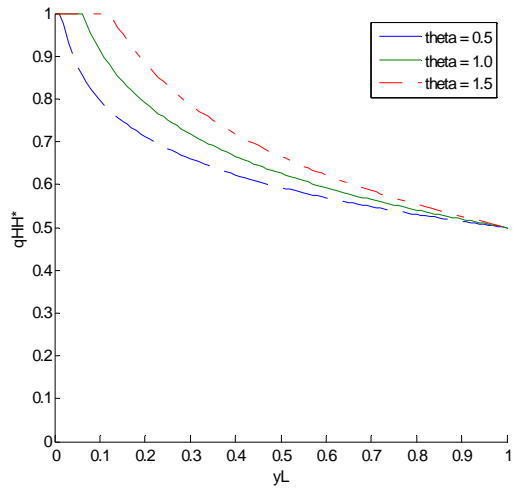


Figure 2: q_{HH}^* as a function of y_L for several values of $\theta_H = \theta_L = \frac{1}{2}\theta$.

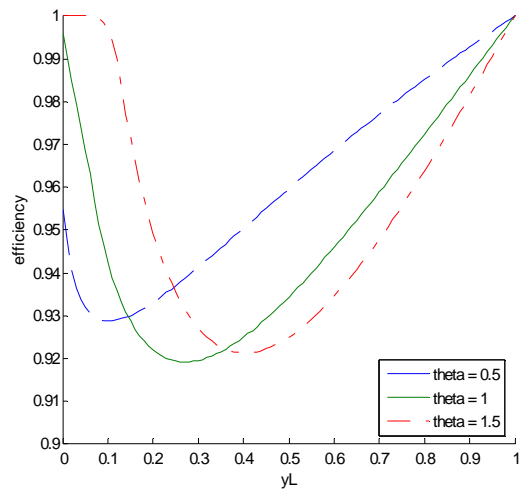


Figure 3: Efficiency of the decentralized equilibrium (Y^*/Y^{**}) as a function of y_L for several values of $\theta_H = \theta_L = \frac{1}{2}\theta$.

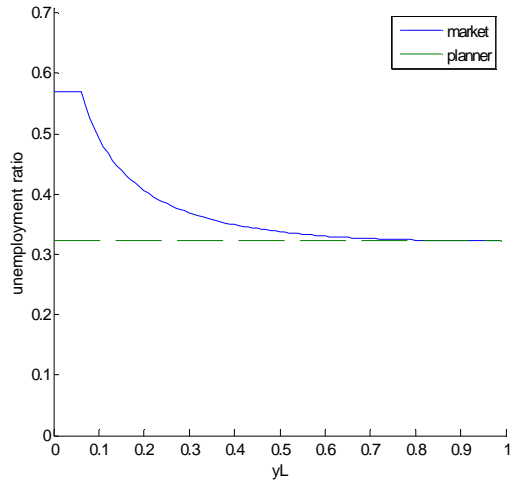


Figure 4: Unemployment ratio as a function of y_L for $\theta_H = \theta_L = 0.5$.

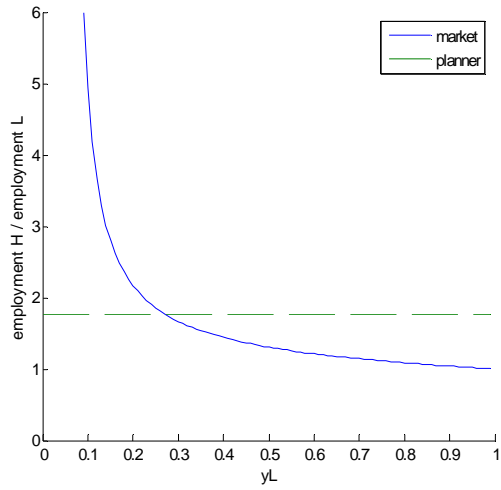


Figure 5: Ratio between employment in the high and low sector as a function of y_L for $\theta_H = \theta_L = \frac{1}{2}$.

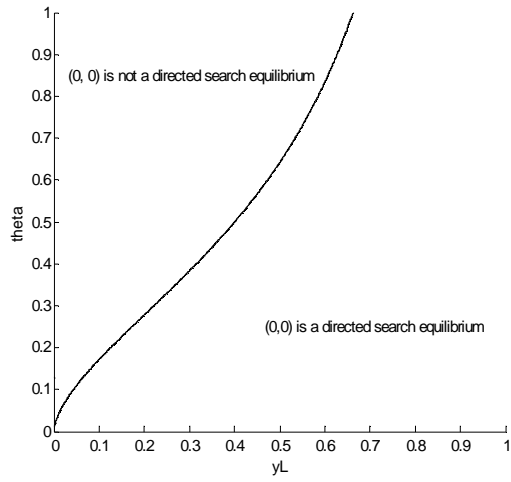


Figure 6: Combinations of y_L and $\theta_H = \theta_L = \frac{1}{2}\theta$ for which $\{w_H = 0, w_L = 0\}$ is a directed search equilibrium.

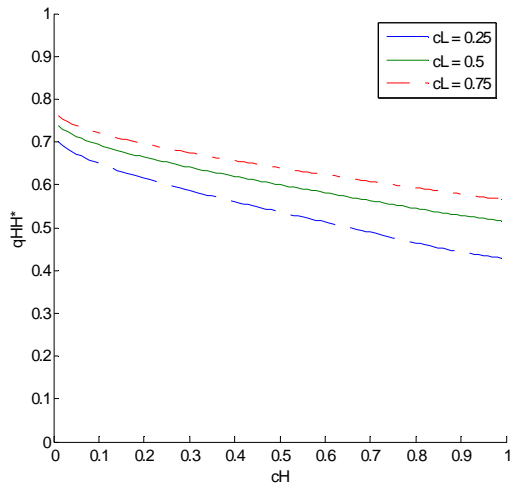


Figure 7: q_{HH}^* as a function of c_H for $\lambda = 0.6$ and several values of c_L .

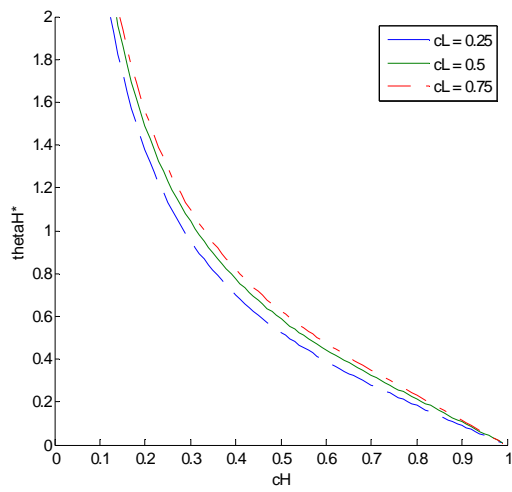


Figure 8: θ_H^* as a function of c_H for $\lambda = 0.6$ and several values of c_L .

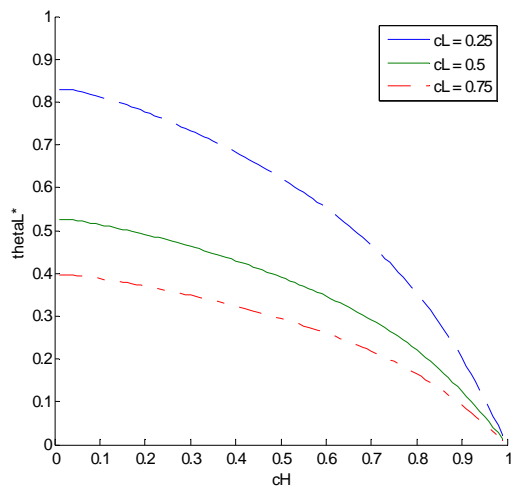


Figure 9: Equilibrium value θ_L^* as a function of c_H for $\lambda = 0.6$ and several values of c_L .

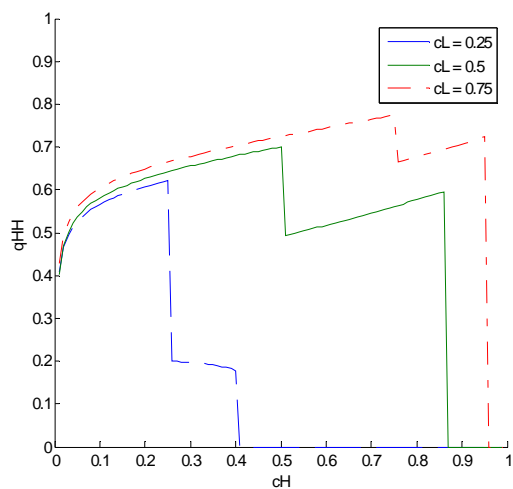


Figure 10: q_{HH}^{**} as a function of c_H for $\lambda = 0.6$ and several values of c_L .

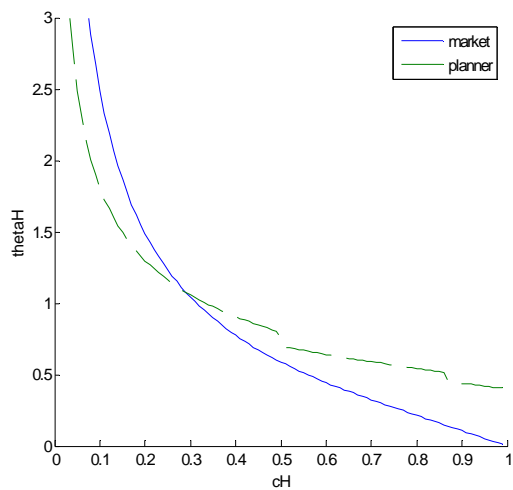


Figure 11: The number of high firms in the market as a function of c_H for $\lambda = 0.6$ and $c_L = 0.5$.

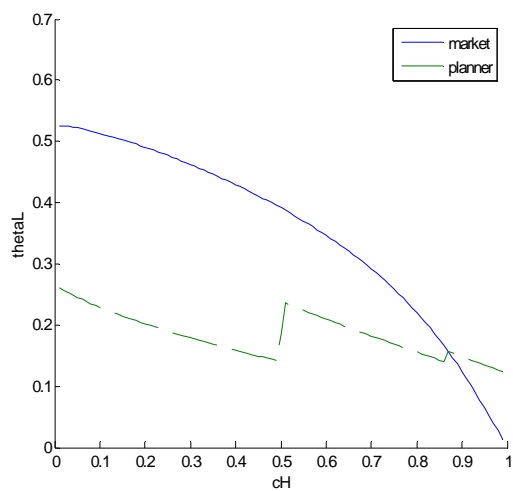


Figure 12: The number of low firms in the market as a function of c_H for $\lambda = 0.6$ and $c_L = 0.5$.

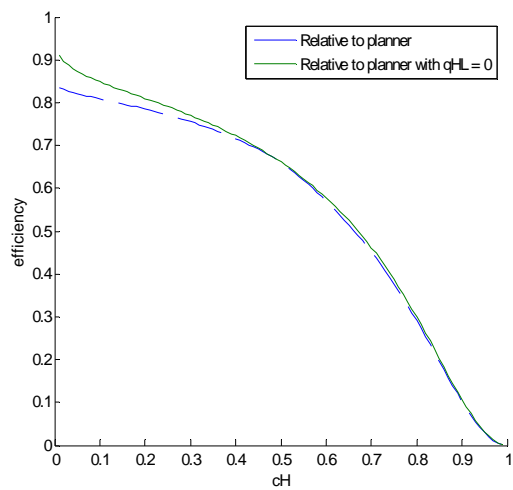


Figure 13: Efficiency of the decentralized equilibrium as a function of c_H for $\lambda = 0.6$ and $c_L = 0.5$.

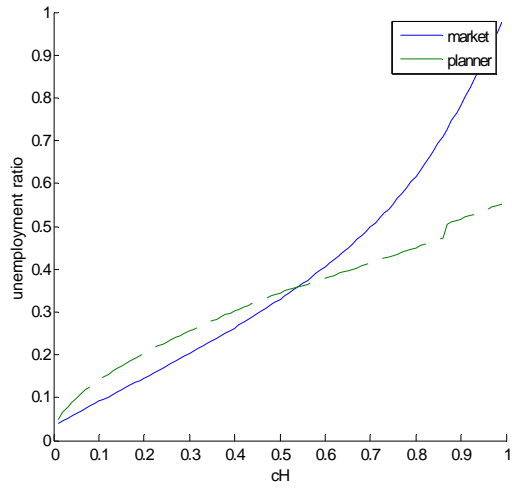


Figure 14: Unemployment ratio for $c_L = 0.5$ and $\lambda = 0.6$ and $c_L = 0.5$.

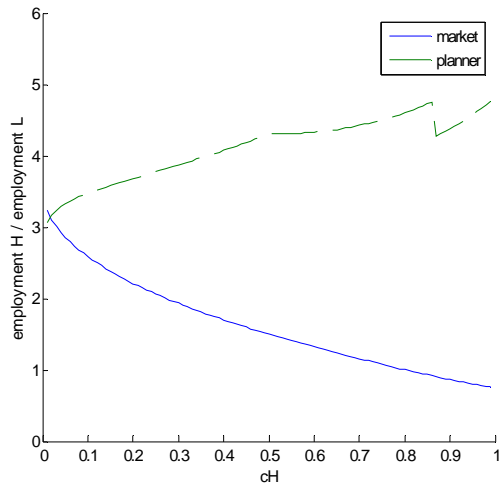


Figure 15: Ratio of employment in high and low sector as a function of c_H for $\lambda = 0.6$ and $c_L = 0.5$.

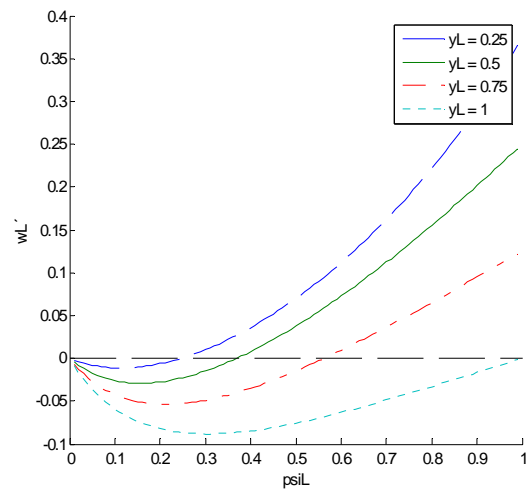


Figure 16: w'_L as a function of ψ_L for several values of y_L . Positive values of w'_L imply that a profitable deviation exists for a low type firm.