The Economic Journal, **121** (*May*) 1–14. Doi: 10.1111/j.1468-0297.2011.02423.x. © 2011 The Author(s). The Economic Journal © 2011 Royal Economic Society. Published by Blackwell Publishing, 9600 Garsington Road, Oxford OX4 2DQ, UK and 350 Main Street, Malden, MA 02148, USA.

Technical Appendix to

ADVERTISING FOR ATTENTION IN A CONSUMER SEARCH MODEL

Marco A. Haan and José L. Moraga-González

Economic Journal, vol. 121, pp. 552–579

Appendix

Formal Statement of Results and Detailed Proofs

Proposition 1. *If a symmetric Nash equilibrium (SNE) exists, advertising levels and prices are given by the following system of equations:*

$$a^*\phi'(a^*) - \frac{p^*}{n}\left\{1 - F(\hat{x})^n - \sum_{k=0}^{n-1}\frac{F(\hat{x})^k[1-F(\hat{x})^{n-k}]}{n-k}\right\} = 0, \tag{1}$$

$$\frac{1-F(p^*/\mu)^n}{n(p^*/\mu)} - \frac{f(\hat{x})}{n}\frac{1-F(\hat{x})^n}{1-F(\hat{x})} + \int_{p^*/\mu}^{\hat{x}} F(\varepsilon)^{n-1}f'(\varepsilon)\,\mathrm{d}\varepsilon = 0. \tag{2}$$

Suppose that F represents the uniform distribution and that ϕ'' is sufficiently large. Then a symmetric equilibrium exists and is unique.

Proof. This proof has four steps. First, we show that the first-order conditions for profit maximisation imply (1) and (2). Second, we show that there exists a pair (p^*,a^*) that satisfies (1) and (2), and that it is unique if $f' \geq 0$. Third, we show that (p^*,a^*) is indeed a Nash equilibrium if we restrict attention to a uniform distribution of matching values, relatively small price defections such that profits are given in (11) of the main text, and sufficiently convex advertising cost functions. Fourth, we show that large defections from (p^*,a^*) are never profitable either.

Step 1. We first derive the expressions (1) and (2) given in the Proposition. Maximising firm i's profits in (11) of the main text with respect to a_i and p_i yields the following first-order conditions:

$$p_i\sum_{k=1}^{n}\frac{\partial\lambda_k^i(a_i,p_i;a^*,p^*)}{\partial a_i} - \phi'(a_i) = 0, \tag{3}$$

$$\sum_{k=1}^{n}\lambda_k^i(a_i,p_i;a^*,p^*) + R(p_i;p^*) + p_i\left[\sum_{k=1}^{n}\frac{\partial\lambda_k^i(a_i,p_i;a^*,p^*)}{\partial p_i} + \frac{\partial R(p_i;p^*)}{\partial p_i}\right] = 0. \tag{4}$$

Using the expression for λ_k^i in (8) of the main text, we obtain

$$\begin{aligned}\frac{\partial\lambda_1^i}{\partial a_i} &= \frac{(n-1)a^*}{[a_i+(n-1)a^*]^2}[1-F(\hat{x}+\Delta)]\\ &\dots\end{aligned} \tag{5}$$

$$
\begin{aligned}
\frac{\partial \lambda_k^i}{\partial a_i} = \Bigg(& \frac{a_i}{a_i + (n-k)a^*} \sum_{\ell=1}^{k-1} \left\{ \frac{-(n-\ell)a^*}{[a_i + (n-\ell)a^*]^2} \prod_{m \neq \ell}^{k-1} \frac{(n-m)a^*}{a_i + (n-m)a^*} \right\} \\
& + \frac{(n-k)a^*}{[a_i + (n-k)a^*]^2} \prod_{\ell=1}^{k-1} \frac{(n-\ell)a^*}{a_i + (n-\ell)a^*} \Bigg) F(\hat{x})^{k-1} [1 - F(\hat{x} + \Delta)]
\end{aligned}
\tag{6}
$$

$$\cdots$$

$$
\frac{\partial \lambda_n^i}{\partial a_i} = \sum_{\ell=1}^{n-1} \left\{ \frac{-(n-\ell)a^*}{[a + (n-\ell)a^*]^2} \prod_{m \neq \ell}^{n-1} \frac{(n-m)a^*}{a + (n-m)a^*} \right\} F(\hat{x})^{n-1} [1 - F(\hat{x} + \Delta)]. \tag{7}
$$

In symmetric equilibrium we have

$$
\frac{\partial \lambda_1^i}{\partial a_i} = \frac{n-1}{n^2 a^*} [1 - F(\hat{x})] \tag{8}
$$

$$\cdots$$

$$
\begin{aligned}
\frac{\partial \lambda_k^i}{\partial a_i} = \Bigg\{ & \frac{1}{n-k+1} \sum_{\ell=1}^{k-1} \left[\frac{-(n-\ell)}{(n-\ell+1)^2 a^*} \prod_{m \neq \ell}^{k-1} \frac{n-m}{n-m+1} \right] \\
& + \frac{n-i}{(n-i+1)^2 a^*} \prod_{\ell=1}^{k-1} \frac{n-\ell}{n-\ell+1} \Bigg\} F(\hat{x})^{k-1} [1 - F(\hat{x})]
\end{aligned}
\tag{9}
$$

$$\cdots$$

$$
\frac{\partial \lambda_n^i}{\partial a_i} = \sum_{\ell=1}^{n-1} \left[\frac{-(n-\ell)}{(n-\ell+1)^2 a^*} \prod_{m \neq \ell}^{n-1} \frac{n-m}{n-m+1} \right] F(\hat{x})^{n-1} [1 - F(\hat{x})]. \tag{10}
$$

Note that

$$
\prod_{\ell=1}^{k-1} \frac{n-\ell}{n+1-\ell} = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \cdots \cdot \frac{n+1-k}{n+2-k} = \frac{n+1-k}{n}, \tag{11}
$$

which allows us to write

$$
\frac{\partial \lambda_k^i}{\partial a_i} = \frac{1}{n a^*} \left(\frac{n-k}{n-k+1} - \sum_{\ell=1}^{k-1} \frac{1}{n-\ell+1} \right) F(\hat{x})^{k-1} [1 - F(\hat{x})]. \tag{12}
$$

Moreover, we note that

$$
\sum_{k=1}^{n} \lambda_k(a^*, p^*) = \frac{1}{n} [1 - F(\hat{x})^n]. \tag{13}
$$

Using these derivations and the expression for $R(p^*)$ in (10) of the main text, the first order conditions (3) and (4) can be rewritten as:

$$
p^* \sum_{k=1}^{n} \frac{1}{n a^*} \left(\frac{n-k}{n-k+1} - \sum_{\ell=1}^{k-1} \frac{1}{n-\ell+1} \right) F(\hat{x})^{k-1} [1 - F(\hat{x})] - \phi'(a^*) = 0, \tag{14}
$$

$$
\begin{aligned}
&\frac{1-F(\hat{x})^n}{n} + \int_{\frac{p^*}{\mu}}^{\hat{x}} F(\varepsilon)^{n-1} f(\varepsilon)\mathrm{d}\varepsilon - \frac{p^*}{\mu}\frac{f(\hat{x})}{n}\frac{1-F(\hat{x})^n}{1-F(\hat{x})} \\
&\qquad - \frac{p^*}{\mu}\left[\int_{\frac{p^*}{\mu}}^{\hat{x}} (n-1)F(\varepsilon)^{n-2} f(\varepsilon)^2 \mathrm{d}\varepsilon + F\left(\frac{p^*}{\mu}\right)^{n-1} f\left(\frac{p^*}{\mu}\right) - F(\hat{x})^{n-1} f(\hat{x})\right] = 0. \qquad (15)
\end{aligned}
$$

It is now readily seen that using the integration by parts formula in (15) yields (2).

To see that (14) implies (1), denote

$$
C_k \equiv \frac{n-k}{n-k+1} - \sum_{\ell=1}^{k-1} \frac{1}{n-\ell+1}, \qquad (16)
$$

so we can rewrite (14) as

$$
a^*\phi'(a^*) = \frac{p^*}{n}[1-F(\hat{x})] \sum_{k=1}^{n} C_k F(\hat{x})^{k-1}. \qquad (17)
$$

Note that

$$
\begin{aligned}
C_k - C_{k-1} &= \left(\frac{n-k}{n-k+1} - \sum_{\ell=1}^{k-1} \frac{1}{n-\ell+1}\right) - \left(\frac{n-k+1}{n-k+2} - \sum_{\ell=1}^{k-2} \frac{1}{n-\ell+1}\right) \\
&= \frac{n-k}{n-k+1} - \frac{1}{n-k+2} - \frac{n-k+1}{n-k+2} = \frac{-1}{n-k+1}.
\end{aligned} \qquad (18)
$$

From (16), we have $C_1 = (n-1)/n$ so by induction

$$
C_k = \frac{n-1}{n} - \sum_{\ell=1}^{k-1} \frac{1}{n-\ell}. \qquad (19)
$$

Substituting this back into (17), we obtain

$$
\begin{aligned}
a^*\phi'(a^*) &= \frac{p^*}{n}[1-F(\hat{x})] \sum_{k=1}^{n} \left(\frac{n-1}{n} - \sum_{\ell=1}^{k-1} \frac{1}{n-\ell}\right) F(\hat{x})^{k-1} \\
&= \frac{p^*}{n}[1-F(\hat{x})] \left[\frac{n-1}{n} \sum_{k=1}^{n} F(\hat{x})^{k-1} - \sum_{k=1}^{n}\sum_{\ell=1}^{k-1} \frac{1}{n-\ell} F(\hat{x})^{k-1}\right] \\
&= \frac{p^*}{n}[1-F(\hat{x})] \left\{\frac{n-1}{n} \sum_{k=1}^{n} F(\hat{x})^{k-1} - \sum_{\ell=1}^{n-1} \left[\frac{1}{n-\ell} \sum_{k=\ell+1}^{n} F(\hat{x})^{k-1}\right]\right\} \\
&= \frac{p^*}{n}[1-F(\hat{x})] \left\{\frac{n-1}{n} \sum_{k=0}^{n-1} F(\hat{x})^{k} - \sum_{\ell=1}^{n-1} \left[\frac{1}{n-\ell} \sum_{k=\ell}^{n-1} F(\hat{x})^{k}\right]\right\},
\end{aligned} \qquad (20)
$$

which can be further simplified to

$$
\begin{aligned}
a^*\phi'(a^*) &= \frac{p^*}{n}[1-F(\hat{x})] \left[\frac{n-1}{n}\frac{1-F(\hat{x})^n}{1-F(\hat{x})} - \sum_{\ell=1}^{n-1} \left(\frac{1}{n-\ell}\right) \frac{F(\hat{x})^{\ell} - F(\hat{x})^n}{1-F(\hat{x})}\right] \\
&= \frac{p^*}{n}\left\{\frac{n-1}{n}[1-F(\hat{x})^n] - \sum_{\ell=1}^{n-1} \left(\frac{1}{n-\ell}\right) F(\hat{x})^{\ell}\left[1-F(\hat{x})^{n-\ell}\right]\right\},
\end{aligned} \qquad (21)
$$

which gives (1).

Step 2. We now show that there exists a pair (p^*, a^*) that satisfies the system of equations (1) and (2). By inspection of (1), it is immediately clear that for any p^* there is a unique a^* that accompanies p^*. To see that such a^* is non-negative, consider the expression

$$A \equiv 1 - F(\hat{x})^n - \sum_{k=0}^{n-1} \frac{F(\hat{x})^k [1 - F(\hat{x})^{n-k}]}{n-k} \tag{22}$$

and notice that

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}\hat{x}} &= -nF(\hat{x})^{n-1} f(\hat{x}) + F(\hat{x})^{n-1} f(\hat{x}) \\ &\quad - \sum_{k=1}^{n-1} \frac{kF(\hat{x})^{k-1}[1 - F(\hat{x})^{n-k}] - (n-k)F(\hat{x})^k F(\hat{x})^{n-k-1}}{n-k} f(\hat{x}) \\ &= -nF(\hat{x})^{n-1} f(\hat{x}) + F(\hat{x})^{n-1} f(\hat{x}) - \sum_{k=1}^{n-1} \frac{kF(\hat{x})^{k-1}[1 - F(\hat{x})^{n-k}]}{n-k} f(\hat{x}) + \sum_{k=1}^{n-1} F(\hat{x})^{n-1} f(\hat{x}) \\ &= -\sum_{k=1}^{n-1} \frac{kF(\hat{x})^{k-1}[1 - F(\hat{x})^{n-k}]}{n-k} f(\hat{x}) < 0. \end{aligned} \tag{23}$$

Therefore, A is monotonically decreasing in $\hat{x}$ and since $\lim_{\hat{x} \to 1} A = 0$ we conclude a^* is non-negative.

Consider now (2). To study the existence of a solution in p^*, it is useful to rewrite it as follows:

$$\frac{1 - F(p^*/\mu)^n}{np^*/\mu} = \frac{f(\hat{x})}{n} \frac{1 - F(\hat{x})^n}{1 - F(\hat{x})} - \int_{p^*/\mu}^{\hat{x}} F(\varepsilon)^{n-1} f'(\varepsilon) d\varepsilon. \tag{24}$$

Note that the RHS of (24) is finite when $p^* \to 0$. The LHS is a positive-valued function that decreases monotonically in p^*. Moreover, when $p^* \to 0$ the LHS goes to ∞. Hence, for $p^* \to 0$ the LHS is larger than the RHS. If $p^* \to \mu\hat{x}$, the LHS is smaller than the RHS if and only if $1 - F(\hat{x}) < \hat{x} f(\hat{x})$. Since $\hat{x} > p^{\mathrm{m}}/\mu > p^*/\mu$ and by definition $1 - F(p^{\mathrm{m}}/\mu) - (p^{\mathrm{m}}/\mu) f(p^{\mathrm{m}}/\mu) = 0$, logconcavity of monopoly profits implies that this condition always holds. With the LHS larger that the RHS at $p^* \to 0$, but smaller at $p^* \to \mu\hat{x}$, continuity implies that there must be at least one $p^* \in (0, \mu\hat{x})$ such that (24) is satisfied. If we assume also that $f' \geq 0$, we have that the RHS is strictly increasing in p^*, in which case (24) has a unique solution.

Step 3. In step 2, we established that there exists a pair (a^*, p^*) that solves (1) and (2). Yet, that does not immediately imply that such an (a^*, p^*) is a SNE. For this to be the case, we need to make sure the payoff function of a firm i is globally quasi-concave on its domain. The next three claims show this. Let D denote the domain of firm i's payoff function. Note that $D \equiv \{(a_i, p_i) \in [0, \infty) \times (0, p^m)\}$ and consider the following partition of the domain $D \equiv D_1 \cup D_2 \cup D_3$ where $D_1 \equiv \{(a_i, p_i) \in (0, \infty) \times (0, \mu F^{-1}(1) - \mu\hat{x} + p^*)\}$, $D_2 \equiv \{(a_i, p_i) \in [0, \infty) \times [\mu F^{-1}(1) - \mu\hat{x} + p^*, p^{\mathrm{m}})\}$ and $D_3 \equiv \{(a_i, p_i) \in \{0\} \times (0, p^{\mathrm{m}})\}$. On the set D_1, the payoff to firm i, $\Pi_i(a_i, p_i; a^*, p^*)$, is given by equation (11) in the main text.[1]

Claim 1. *On D_1, the function $\Pi_i(a_i, p_i; a^*, p^*)$ is strictly concave in a_i.*

To see this, define the function

---

[1] Deviations for which $p_i \geq \mu F^{-1}(1) - \mu\hat{x} + p^*$ are special because in those situations firm i would only sell to consumers who have walked away from all other rivals; we treat these cases later in step 4.

$$
\begin{aligned}
y_n(a_i, a^*) \equiv{}& \frac{a_i}{a_i + (n-1)a^*} + \frac{(n-1)a^*}{a_i + (n-1)a^*}\frac{a_i}{a_i + (n-2)a^*}F(\hat{x}) \\
&+ \sum_{k=3}^{n} \frac{a_i}{a_i + (n-k)a^*} \prod_{\ell=1}^{k-1} \frac{(n-\ell)a^*}{a_i + (n-\ell)a^*} F(\hat{x})^{k-1},
\end{aligned}
\tag{25}
$$

so we can write the payoff given in (11) of the main text as

$$\Pi_i(a_i, p_i; a^*, p^*) = p_i y_n(a_i, a^*)[1 - F(\hat{x} + \Delta)] + p_i R(p_i; p^*) - \phi(a_i). \tag{26}$$

Note that y_n reflects the probability that firm i will be visited given that all other firms stick to the candidate SNE price and advertising level. Also note that

$$y_2(a_i, a^*) = \frac{a_i}{a_i + a^*} + \frac{a^*}{a_i + a^*}F(\hat{x}), \tag{27}$$

and, moreover

$$y_{k+1}(a_i, a^*) = \frac{a_i}{a_i + ka^*} + \frac{ka^*}{a_i + ka^*}F(\hat{x})y_k, \tag{28}$$

for any $k > 2$. Taking the derivative of y_2 with respect to a_i:

$$y_2'(a_i, a^*) = \frac{a^*}{(a_i + a^*)^2}[1 - F(\hat{x})] > 0. \tag{29}$$

Hence $y_2(a_i, a^*)$ is strictly increasing in a_i. For y_{k+1}', we can write

$$y_{k+1}'(a_i, a^*) = \frac{ka^*}{(a_i + ka^*)^2}[1 - F(\hat{x})y_k] + F(\hat{x})\left(\frac{ka^*}{a_i + ka^*}\right)y_k'. \tag{30}$$

Note that $F(\hat{x})y_k(a_i, a^*) < 1$. Therefore, $y_k'(a_i, a^*) > 0$ suffices for this expression to be positive. We already know that $y_2'(a_i, a^*) > 0$. Hence, $y_3'(a_i, a^*) > 0$. By induction, $y_{k+1}'(a_i, a^*) > 0$ for any k.

For the second derivative, we have

$$y_2''(a_i, a^*) = \frac{-2a^*[1 - F(\hat{x})]}{(a_i + a^*)^3} < 0. \tag{31}$$

and

$$y_{k+1}''(a_i, a^*) = \frac{-2ka^*}{(a_i + ka^*)^3}[1 - F(\hat{x})y_k] + F(\hat{x})\left[\frac{ka^*}{a_i + ka^*}y_k'' - 2\frac{ka^*}{(a_i + ka^*)^2}y_k'\right]. \tag{32}$$

Note that $F(\hat{x})y_k(a_i, a^*) < 1$. Hence, sufficient for this expression to be negative is that $y_k''(a_i, a^*) < 0$ and $y_k'(a_i, a^*) > 0$. But we already know that this holds for $k = 2$. Hence, from this expression, it also holds for $k = 3$. Induction then implies that it holds for any k. With $y_{k+1}''(a_i, a^*) < 0$, we immediately have that $\partial^2\Pi(\cdot)/\partial a_i^2 < 0$ for any (weakly) convex function $\phi(a_i)$.

CLAIM 2. *On D_1, the function $\Pi_i(a_i, p_i; a^*, p^*)$ is not necessarily quasi-concave in p_i. However, when F represents the uniform distribution, then $\Pi_i(a_i, p_i; a^*, p^*)$ is strictly concave in p_i.*

To see that $\Pi_i(a_i,p_i;a^*,p^*)$ is not generally quasi-concave in p_i, consider the case in which $\hat{x} \to 1$, so search cost s go to zero. In that case our model gives Perloff and Salop (1985). From Caplin and Nalebuff (1991), we know that the payoff function

$$p_i \int_{p_i/\mu}^{1} F(\varepsilon - \Delta)^{n-1} f(\varepsilon)\mathrm{d}\varepsilon, \tag{33}$$

is quasi-concave if the density f is log-concave. However, with strictly positive search costs ($\hat{x} < 1$), our payoff function equals a summation of functions of p_i. This sum may not be quasi-concave in p_i, even if every summand is quasi-concave. In fact, if one sets $a_i = a^*$ above, our model gives Anderson and Renault (1999) and, as they show, with positive search costs stronger conditions are needed for the payoff to be quasi-concave (see their Appendix B). We therefore focus on the case where F is the uniform distribution. In that case we have

$$\frac{\partial^2 \Pi_i(a_i, p_i; a^*, p^*)}{\partial p_i^2} = -\frac{2}{\mu}\sum_{k=1}^{n}\frac{a_i}{a_i + (n-k)a^*}\prod_{\ell=1}^{k-1}\frac{(n-\ell)a^*}{a_i + (n-\ell)a^*}\hat{x}^{k-1} < 0, \tag{34}$$

which implies that $\Pi_i(a_i,p_i;a^*,p^*)$ is strictly concave in p_i.

Claim 3. *When F is the uniform distribution, and when ϕ'' is sufficiently large, the function $\Pi_i(a_i, p_i; a^*, p^*)$ is globally strictly concave on D_1.*

The Hessian of Π_i is given by the matrix

$$\mathbf{H} \equiv \begin{pmatrix} \dfrac{\partial^2 \Pi_i}{\partial p_i^2} & \dfrac{\partial^2 \Pi_i}{\partial p_i \partial a_i} \\ \dfrac{\partial^2 \Pi_i}{\partial p_i \partial a_i} & \dfrac{\partial^2 \Pi_i}{\partial a_i^2} \end{pmatrix}. \tag{35}$$

We already know that $\partial^2\Pi(\cdot)/\partial p_i^2 < 0$ and $\partial^2\Pi(\cdot)/\partial a_i^2 < 0$. Therefore, it suffices to show that the determinant of $\mathbf{H}$ is strictly positive. That is $[\partial^2\Pi(\cdot)/\partial p_i^2][\partial^2\Pi(\cdot)/\partial a_i^2] - [\partial^2\Pi(\cdot)/\partial p_i \partial a_i]^2 > 0$, which holds whenever $\partial^2\Pi(\cdot)/\partial a_i^2$ is sufficiently negative.

Claims 1, 2 and 3 together imply that there does not exist any profitable deviation from (a^*, p^*) in the set D_1 provided that matching values are uniformly distributed and the advertising cost function is sufficiently convex. To complete the proof, we now study deviations outside the set D_1.

Step 4. Consider now deviations to pairs (a_i, p_i) in the sets D_2 and D_3 defined above, that is, we need to make sure that a firm i has no interest in deviating by charging a price such that $1 - F(\hat{x} + \Delta) = 0$. In that case no consumer would ever stop searching at firm i and the deviant firm would only sell to the consumers who happen to find no acceptable product elsewhere. Deviating profits for this situation would be

$$\Pi_i(a_i, p_i; a^*, p^*) = p_i \int_{p_i/\mu}^{1} F(\varepsilon - \Delta)^{n-1} f(\varepsilon)\ \mathrm{d}\varepsilon - \phi(a_i). \tag{36}$$

By monotonicity of this payoff, it is clear that the deviant firm would find it optimal to accompany the deviating price with an advertising effort $= 0$.[2] Because of log-concavity of f, this profits expression is quasi-concave in p_i (Caplin and Nalebuff, 1991). Taking the derivative with respect to p_i yields

$$\int_{p_i/\mu}^{1} F(\varepsilon - \Delta)^{n-1} f(\varepsilon)\mathrm{d}\varepsilon - \frac{p_i}{\mu} F\left(\frac{p^*}{\mu}\right)^{n-1} f\left(\frac{p_i}{\mu}\right) - (n-1)\frac{p_i}{\mu}\int_{p_i/\mu}^{1} F(\varepsilon - \Delta)^{n-2} f(\varepsilon - \Delta) f(\varepsilon)\ \mathrm{d}\varepsilon. \tag{37}$$

Setting $p_i = p^*$ in this expression we have:

---

[2] Likewise, notice that if the deviating firm sets an advertising effort $= 0$, the firm would be visited last and its profit would be similar to that in (36).

$$\int_{p^*/\mu}^{1} F(\varepsilon)^{n-1} f(\varepsilon)\mathrm{d}\varepsilon - \frac{p^*}{\mu}\left[F(p^*/\mu)^{n-1} f(p^*/\mu) + (n-1)\int_{p^*/\mu}^{1} F(\varepsilon)^{n-2} f(\varepsilon)^2 \mathrm{d}\varepsilon\right] = \frac{1-F(p^*/\mu)^n}{n} - \frac{p^*}{\mu}\left[f(1) - \int_{p^*/\mu}^{1} F(\varepsilon)^{n-1} f'(\varepsilon)\,\mathrm{d}\varepsilon\right]. \tag{38}$$

where the last equality follows from integration by parts. This expression is exactly the limit of the first order condition in (2) when $\hat{x} \to 1$. We will show later in the proof of Proposition 2 that the expression

$$\frac{f(\hat{x})}{n}\frac{1-F(\hat{x})^n}{1-F(\hat{x})} - \int_{p^*/\mu}^{\hat{x}} F(\varepsilon)^{n-1} f'(\varepsilon)\; d\varepsilon. \tag{39}$$

is increasing in $\hat{x}$. This implies that (38) is negative and therefore the profits expression in (36) is decreasing at $p_i = p^*$. This fact along with the quasi-concavity of the expression in (36) implies that deviating profits are monotonically decreasing in p_i, for all $p_i \in [\tilde{p}_i, p^m]$, where $\tilde{p}_i$ is the solution to $1 - F[\hat{x} + (\tilde{p}_i - p^*)/\mu] = 0$. As a result, deviating to a price above p^* is not profitable.

Taken together, steps 1, 2, 3 and 4 establish the proposition.

PROPOSITION 2.

(*1*) *An increase in the marginal cost of advertising has no effect on equilibrium prices* p^* *and lowers the equilibrium number of ads* a^*.

(*2*) *If the density of match values is non-decreasing,* $f' \geq 0$, *an increase in search costs* s *raises equilibrium price* p^* *and raises the equilibrium number of ads* a^*.

(*3*) *For* μ *sufficiently large, an increase in* μ *raises equilibrium price* p^*. *When* $n = 2$ *and* F *is the uniform distribution, the equilibrium number of ads* a^* *decreases in* μ.

(*4*) *As the number of firms approaches infinity, per-firm advertising goes to zero and aggregate industry advertising converges to the constant* $\{\mu[1 - F(\hat{x})]\}/[f(\hat{x})\phi'(0)]$. *If the number of firms is sufficiently low and match values are uniformly distributed, an increase in* n *increases per-firm advertising.*

Proof. (1) The price result follows straightforwardly from the equilibrium condition (2), which does not depend on advertising costs. For the advertising result, consider the equilibrium condition (1) and note that a change in advertising costs should leave the product $a^*\phi'(a^*)$ constant. Consider two advertising cost functions ϕ_1 and ϕ_2, with $\phi_1'(a) > \phi_2'(a)$ for all a. Equilibrium requires $a_1^*\phi_1'(a_1^*) = a_2^*\phi_2'(a_2^*)$. As $\phi_1' > \phi_2'$, we require $a_1^*\phi_1'(a_1^*) < a_2^*\phi_1'(a_2^*)$. Convexity of ϕ_1 implies that $a\phi_1'(a)$ is strictly increasing in a, hence equilibrium requires $a_1^* < a_2^*$.

(2) We start with the price result. Building on the proof of Proposition 1, the equilibrium price is given by the solution to (24). In this equation the effects of higher search costs are manifested only through changes in $\hat{x}$. The LHS of (24) decreases in p^* and does not depend on $\hat{x}$. The RHS is non-decreasing in p^* for any distribution that has $f' \geq 0$. Taking the derivative of the RHS of (24) with respect to $\hat{x}$ yields:

$$\frac{\{f'(\hat{x})[1-F(\hat{x})^n] - nF(\hat{x})^{n-1}f^2(\hat{x})\}[1-F(\hat{x})] + f(\hat{x})^2[1-F(\hat{x})^n]}{n[1-F(\hat{x})]^2} - F(\hat{x})^{n-1}f'(\hat{x}). \tag{40}$$

Collecting terms the expression in (40) can be rewritten as:

$$\frac{1}{n}\left[f'(\hat{x}) + \frac{f^2(\hat{x})}{1-F(\hat{x})}\right]\left[\frac{1-F(\hat{x})^n}{1-F(\hat{x})} - nF(\hat{x})^{n-1}\right]. \tag{41}$$

The first term is positive because of log concavity of $1-F$. The second term is also positive because it equals $\sum_{k=0}^{n-1}[F(\hat{x})^k - F(\hat{x})^{n-1}]$ and F is a distribution function. We thus have that the RHS of (24) is increasing in $\hat{x}$ and therefore decreasing in s.

We now take the advertising result. Rewrite a^* as

$$a^*\phi'(a^*) = \frac{p^*A}{n}, \tag{42}$$

where A is given in (22). Taking the derivative of $a^*\phi'(a^*)$ with respect to $\hat{x}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\hat{x}}[a^*\phi'(a^*)] = \frac{A}{n}\frac{\mathrm{d}p^*}{\mathrm{d}\hat{x}} + \frac{p^*}{n}\frac{\mathrm{d}A}{\mathrm{d}\hat{x}}. \tag{43}$$

Since ϕ is convex, we need $a^*\phi'(a^*)$ to be decreasing in $\hat{x}$. But this is true since we know already that $\mathrm{d}p^*/\mathrm{d}\hat{x} < 0$ and in Proposition 1 we have shown that $\mathrm{d}A/\mathrm{d}\hat{x} < 0$.

(3) For the price result, we apply the implicit function theorem to (2). Let $\Gamma(\cdot)$ denote the LHS of (2). Noting that $\hat{x}$ depends positively on μ, we have

$$\frac{\mathrm{d}p^*}{\mathrm{d}\mu} = -\frac{\dfrac{\partial\Gamma}{\partial\mu} + \dfrac{\partial\Gamma}{\partial\hat{x}}\dfrac{\partial\hat{x}}{\partial\mu}}{\dfrac{\partial\Gamma}{\partial p^*}} = \frac{p^*}{\mu} + \frac{-\dfrac{\partial\Gamma}{\partial\hat{x}}\dfrac{\partial\hat{x}}{\partial\mu}}{\dfrac{\partial\Gamma}{\partial p^*}}. \tag{44}$$

Note now that $-\partial\Gamma/\partial\hat{x}$ equals (41) while

$$\frac{\partial\hat{x}}{\partial\mu} = \frac{s}{\mu^2[1-F(\hat{x})]}. \tag{45}$$

In addition,

$$\frac{\partial\Gamma}{\partial p^*} = -\frac{nF(p^*/\mu)^{n-1}f(p^*/\mu)(p^*/\mu) + [1-F(p^*/\mu)^n]}{np^{*2}/\mu} - \frac{1}{\mu}F(p^*/\mu)^{n-1}f'(p^*/\mu). \tag{46}$$

Consider the case where μ is large ($\mu \to \infty$). In such a case, $\hat{x} \to 1$ and $F(\hat{x}) \to 1$ so the ratio p^*/μ is finite since it is the solution to the following equation

$$\frac{1-F(p^*/\mu)^n}{n(p^*/\mu)} - f(1) + \int_{p^*/\mu}^{1} F(\varepsilon)^{n-1}f'(\varepsilon)\,\mathrm{d}\varepsilon = 0. \tag{47}$$

Moreover, since $\lim_{F\to 1}(1-F^n)/(1-F) = n$

$$-\lim_{\mu\to\infty}\frac{\partial\Gamma}{\partial\hat{x}} = \lim_{\mu\to\infty}\frac{1}{n}\frac{f^2(\hat{x})}{[1-F(\hat{x})]}\left[\frac{(1-F(\hat{x})^n}{1-F(\hat{x})} - nF(\hat{x})^{n-1}\right]. \tag{48}$$

Using the L'Hopital rule we obtain

$$\lim_{\mu\to\infty}\frac{\partial\hat{x}}{\partial\mu} = \lim_{\mu\to\infty}\frac{s}{\mu^2[1-F(\hat{x})]} = \lim_{\mu\to\infty}\frac{-2s/\mu^3}{-f(\hat{x})\partial\hat{x}/\partial\mu} = \lim_{\mu\to\infty}\frac{2[1-F(\hat{x})]}{\mu f(\hat{x})}. \tag{49}$$

Therefore

$$-\lim_{\mu\to\infty}\frac{\partial\Gamma}{\partial\hat{x}}\frac{\partial\hat{x}}{\partial\mu} = \lim_{\mu\to\infty}\frac{2}{n}\frac{f(\hat{x})}{\mu}\left[\frac{1-F(\hat{x})^n}{1-F(\hat{x})} - nF(\hat{x})^{n-1}\right] = 0. \tag{50}$$

Since

$$\lim_{\mu\to\infty}\frac{\partial\Gamma}{\partial p^*} = -\infty, \tag{51}$$

we conclude that $\lim_{\mu\to\infty}\mathrm{d}p^*/\mathrm{d}\mu = \lim_{\mu\to\infty}(p^*/\mu) > 0$.

To prove the advertising result, we first solve the model when $n = 2$ and F is the uniform distribution. This gives

$$p^* = \frac{\mu}{2}\left[\sqrt{5+\hat{x}(2+\hat{x})} - 1 - \hat{x}\right], \tag{52}$$

$$a^*\phi'(a^*) = \frac{p^*(1-\hat{x})^2}{4}. \tag{53}$$

where $\hat{x} = 1 - \sqrt{2s/\mu}$. We now have

$$\begin{aligned}\frac{4}{1-\hat{x}}\frac{\mathrm{d}[a^*\phi'(a^*)]}{\mathrm{d}\mu} &= (1-\hat{x})\frac{\mathrm{d}p^*}{\mathrm{d}\mu} - 2p^*\frac{\mathrm{d}\hat{x}}{\mathrm{d}\mu} = (1-\hat{x})\left(\frac{\partial p^*}{\partial \mu} + \frac{\partial p^*}{\partial \hat{x}}\frac{\mathrm{d}\hat{x}}{\mathrm{d}\mu}\right) - 2p^*\frac{\mathrm{d}\hat{x}}{\mathrm{d}\mu}\\ &= (1-\hat{x})\frac{p^*}{\mu} + \left[\frac{\partial p^*}{\partial \hat{x}}(1-\hat{x}) - 2p^*\right]\frac{\mathrm{d}\hat{x}}{d\mu}\\ &= (1-\hat{x})\frac{p^*}{\mu} - \left[\frac{p^*(1-\hat{x})}{\sqrt{5+\hat{x}(2+\hat{x})}} + 2p^*\right]\frac{1-\hat{x}}{2\mu}\\ &= -(1-\hat{x})\frac{p^*}{\mu}\left[\frac{1-\hat{x}}{2\sqrt{5+\hat{x}(2+\hat{x})}}\right] < 0. \end{aligned}\tag{54}$$

Since ϕ is convex, the result follows.

(4) Let (a_n, p_n) be the solution to the first order conditions (1) and (2) when the number of firms is n. We first prove that $a_n \to 0$ as $n \to \infty$. First note that $a_n \to 0$ if and only if $a_n\phi'(a_n) \to 0$. From (42), we have

$$\lim_{n\to\infty} a_n\phi'(a_n) = \lim_{n\to\infty} p_n \lim_{n\to\infty}\frac{A}{n}. \tag{55}$$

It is easy to see that $\lim_{n\to\infty} p_n = \mu[1-F(\hat{x})]/f(\hat{x})$, which is strictly positive (Wolinsky, 1986). Therefore we need to show that $\lim_{n\to\infty} A/n = 0$. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{A}{n} &= \lim_{n\to\infty}\left\{\frac{1}{n} - \frac{F(\hat{x})^n}{n} - \sum_{k=0}^{n-1}\frac{F(\hat{x})^k[1-F(\hat{x})^{n-k}]}{n(n-k)}\right\}\\ &= -\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{F(\hat{x})^k[1-F(\hat{x})^{n-k}]}{n(n-k)} = -\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{n-k},\end{aligned}\tag{56}$$

where the last equality follows from the fact that $F(\hat{x})^k[1-F(\hat{x})^{n-k}]$ is strictly positive and bounded by 1. Consider the sum $\sum_{k=0}^{n-1} 1/(n-k)$, which can be rewritten as $\sum_{k=1}^{n} 1/k$. It is known that the Euler number γ is given by

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right). \tag{57}$$

Therefore

$$\lim_{n\to\infty}\frac{A}{n} = -\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{n-k} = -\lim_{n\to\infty}\frac{\gamma + \ln n}{n} = 0. \tag{58}$$

For general advertising costs we have $na^* = p^*A/\phi(a^*)$. Arguments as above show that $\lim_{n\to\infty} A = 1$ we have $\lim_{n\to\infty} na^* = \mu[1-F(\hat{x})]/[f(\hat{x})\phi'(0)]$.

Finally we prove that when F is the uniform distribution, an increase in n increases a^* for n sufficiently low. Setting $n = 2$ in (1) yields $a_2\phi'(a_2) = p_2[1 - F(\hat{x})]^2/4$ while setting $n = 3$ in the same first order condition yields $a_3\phi'(a_3) = p_3[1 - F(\hat{x})]^2[4 + 5F(\hat{x})]/18$. Since $a\phi'(a)$ is increasing in a, we have that $a_3 > a_2$ provided that $p_3[4 + 5F(\hat{x})]/9 > p_2/2$. We note first that this inequality clearly holds when F is the uniform distribution and search costs are high, that is, $\hat{x} \to p^{\mathrm{m}}/\mu$. In such a case, $F(p^{\mathrm{m}}/\mu) \to 1/2$ and p_2 and p_3 both approach the monopoly price. Moreover, we note that for $n = 2$ and $n = 3$, it is possible to solve for equilibrium prices. Doing so, tedious calculations reveal that the required inequality is satisfied for the entire range of search costs.

PROPOSITION 3. *Assume the density of match values f' is non-decreasing. Then:*

(*1*) *The profits of a firm increase in search cost s if the search cost s is small enough.*
(*2*) *Consider the family of advertising cost functions $\phi(a) = \alpha a^\gamma/\gamma$, with $\alpha > 0$, $\gamma \geq 1$. Then (i) the profits of a firm increase in search cost s if γ is sufficiently large.*
  (ii) For γ small and sufficiently large search costs s, profits may decrease in s and eventually fall below the profits that firms would make in a frictionless world. In particular, this is true with 2 firms, uniformly distributed matching values and linear or quadratic advertising costs.

Proof. First note that the payoff of a typical firm in symmetric equilibrium is:

$$\Pi_i(a^*, p^*) = \frac{1}{n} p^*[1 - F(p^*/\mu)^n] - \phi(a^*). \tag{59}$$

We are interested in the derivative of Π_i with respect to search cost s. Since $\hat{x}$ decreases in s, we need to study

$$\frac{\mathrm{d}\Pi(\cdot)}{\mathrm{d}\hat{x}} = \frac{\partial\Pi}{\partial p^*}\frac{\mathrm{d}p^*}{\mathrm{d}\hat{x}} + \frac{\partial\Pi}{\partial a^*}\frac{\mathrm{d}a^*}{\mathrm{d}\hat{x}}. \tag{60}$$

Equation (60) shows that search costs do not affect profits directly but via price and advertising efforts. From Proposition 2, we know that $\mathrm{d}p^*/\mathrm{d}\hat{x} < 0$ and $\mathrm{d}a^*/\mathrm{d}\hat{x} < 0$. In equilibrium it is obvious that all firms gain if they all raise their prices, that is, $\partial\Pi/\partial p^* > 0$. This implies that an increase in search costs tends to raise profits because prices increase; however, since $\partial\Pi/\partial a^* = -\phi'(a^*) < 0$, an increase in search costs tends to lower profits because advertising efforts go up. As a result, an increase in search costs operates on profits in two ways that go in opposite directions.

1. To prove this, we first use the first order condition (1), to rewrite (60) as

$$\frac{\mathrm{d}\Pi(\cdot)}{\mathrm{d}\hat{x}} = \left[\frac{\partial\Pi}{\partial p^*} - \phi'(a^*)\frac{\partial a^*}{\partial p^*}\right]\frac{\mathrm{d}p^*}{\mathrm{d}\hat{x}} - \phi'(a^*)\frac{\partial a^*}{\partial \hat{x}}. \tag{61}$$

Second we note that, from (42), we have

$$\frac{\partial a^*}{\partial p^*} = \frac{A}{n[\phi'(a^*) + a^*\phi''(a^*)]} \tag{62}$$

Moreover, from the derivations above in Proposition 2 we have that

$$\frac{\mathrm{d}p^*}{\mathrm{d}\hat{x}} = \frac{-\dfrac{\partial\Gamma}{\partial\hat{x}}}{\dfrac{\partial\Gamma}{\partial p^*}} = \frac{-\dfrac{1}{n}\left[f'(\hat{x}) + \dfrac{f^2(\hat{x})}{1 - F(\hat{x})}\right]\left[\dfrac{1 - F(\hat{x})^n}{1 - F(\hat{x})} - nF(\hat{x})^{n-1}\right]}{\dfrac{nF(p^*/\mu)^{n-1}f(p^*/\mu)(p^*/\mu) + [1 - F(p^*/\mu)^n]}{np^{*2}/\mu} + \dfrac{1}{\mu}F(p^*/\mu)^{n-1}f'(p^*/\mu)}. \tag{63}$$

Consider the case where search costs are very small, that is $\hat{x} \to 1$ and so $F(\hat{x}) \to 1$. In such a case, since $\lim_{F\to 1}(1-F^n)/(1-F) = n$, the numerator of (63) goes to $-f^2(1)(n-1)/2$. When $f' \geq 0$ the denominator is positive so we conclude that $\lim_{\hat{x}\to 1} \mathrm{d}p^*/\mathrm{d}\hat{x}$ is finite and negative. We note now that $\lim_{\hat{x}\to 1} A = 0$ so $\lim_{\hat{x}\to 1} \partial a^*/\partial p^* = 0$. As a result, the first term in the RHS of (61) is a finite negative number when $\hat{x} \to 1$.

Consider now the second term in the RHS of (61). Using (42) again, we have

$$\frac{\partial a^*}{\partial \hat{x}} = \frac{p^*(\partial A/\partial \hat{x})}{n[\phi'(a^*) + a^*\phi''(a^*)]} \tag{64}$$

and substituting (23) in this equation we obtain

$$\lim_{\hat{x}\to 1} \frac{\partial a^*}{\partial \hat{x}} = \lim_{\hat{x}\to 1} \frac{p^*}{n[\phi'(a^*) + a^*\phi''(a^*)]} \lim_{\hat{x}\to 1} \sum_{k=1}^{n-1} \frac{kF(\hat{x})^{k-1}[1 - F(\hat{x})^{n-k}]}{n-k} f(\hat{x}) = 0 \tag{65}$$

for any increasing advertising cost function. As a result we have proven that $\lim_{\hat{x}\to 1} \mathrm{d}\Pi(\cdot)/\mathrm{d}\hat{x}$ equals a finite negative number. Since $\lim_{\hat{x}\to 1} \mathrm{d}\hat{x}/\mathrm{d}s = -\infty$, we conclude profits increase in a neighbourhood of $s = 0$.

2. (*i*) For the family of advertising cost functions $\phi(a)=\alpha a^\gamma$, with $\alpha>0$, $\gamma \geq 1$, it holds that $\phi(a^*)=a^*\phi'(a^*)/\gamma$ so from the equilibrium condition (1) we have $\phi(a^*)-p^*A/(\gamma n)=0$. From this we have

$$-\phi'(a^*)\frac{\mathrm{d}a^*}{\mathrm{d}\hat{x}} = -\frac{1}{\gamma n}\left(\frac{\partial p^*}{\partial \hat{x}} A + p^* \frac{\partial A}{\partial \hat{x}}\right), \tag{66}$$

which goes to zero as $\gamma \to \infty$. As a result, the advertising effect in (60) vanishes and therefore profits increase in *s*.

(*ii*) However, profits need not be increasing in search costs. Here we provide a counter-example. Consider again the case when $n = 2$ and F is the uniform distribution. Equilibrium price and advertising are given in (52) and (53), while profits equal

$$\Pi^* = \frac{p^*}{2}\left[1 - \left(\frac{p^*}{\mu}\right)^2\right] - \phi(a^*) = \frac{p^*}{2}\left[1 - \left(\frac{p^*}{\mu}\right)^2 - \frac{(1-\hat{x})^2}{2\gamma}\right], \tag{67}$$

where *s* ranges from 0 to $\mu/8$ in this case. Setting $\mu = 1$, Figure A1 plots Π^* against search cost for various values of the elasticity of the advertising cost function γ. It can be seen that for relatively low values of γ, profits are non-monotonic in *s*, first increasing and then decreasing.

PROPOSITION 4. *With two firms, a uniform distribution of matching values and asymmetric advertising technologies, we have that the firm that advertises more, sets a lower price:* $a_i^* > a_j^*$ *necessarily implies* $p_i^* < p_j^*$.

Proof. Define the probability that i is visited first as γ. Thus $\gamma \equiv a_i^*/(a_i^* + a_j^*)$. After setting $p_i = p_i^*$, the FOC in prices can then be written as

$$\begin{aligned} h_1(\gamma, p_1^*, p_2^*) &\equiv \gamma\left[1 - \hat{x} - 2p_1^* + p_2^* + \frac{1}{2}(\hat{x}^2 - p_2^{*2})\right] \\ &+ (1-\gamma)\left[\frac{1}{2}(2 - \hat{x} - 3p_1^*)(\hat{x} - p_1^* + p_2^*) + \frac{1}{2}(\hat{x} - p_1^*)p_2^*\right] = 0. \end{aligned} \tag{68}$$

For the other firm we have

$$\begin{aligned} h_2(1-\gamma, p_2^*, p_1^*) &\equiv (1-\gamma)\left[1 - \hat{x} - 2p_2^* + p_1^* + \frac{1}{2}(\hat{x}^2 - p_1^{*2})\right] \\ &+ \gamma\left[\frac{1}{2}(2 - \hat{x} - 3p_2^*)(\hat{x} - p_2^* + p_1^*) + \frac{1}{2}(\hat{x} - p_2^*)p_1^*\right] = 0. \end{aligned} \tag{69}$$

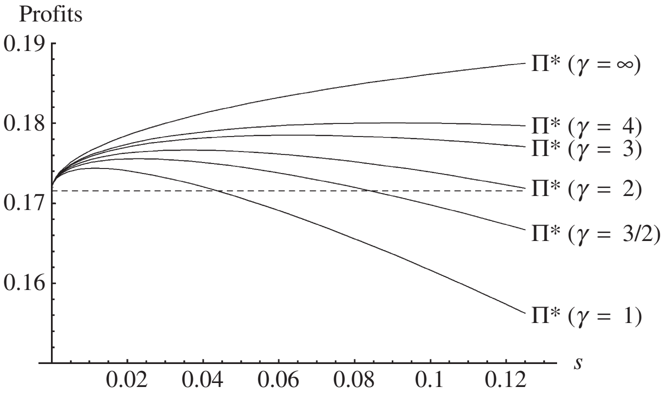

Fig. A1. *Equilibrium Profits* $(n = 2,\ f = 1,\ \phi(a) = \alpha a^{\gamma})$

This implies that equilibrium also requires that

$$h_1(\gamma, p_1^*, p_2^*) = h_2(1-\gamma, p_2^*, p_1^*), \tag{70}$$

which we can rewrite as

$$2\gamma = \frac{4\hat{x} - 4p_1^* + 6p_2^* - 2\hat{x}p_1^* + 4p_1^{*2} - 4p_1^*p_2^* - 2\hat{x}^2 - 2}{4\hat{x} + p_1^* + p_2^* - \hat{x}p_1^* - \hat{x}p_2^* + 2p_1^{*2} + 2p_2^{*2} - 4p_1^*p_2^* - 2\hat{x}^2 - 2}. \tag{71}$$

Now suppose that $2\gamma > 1$ and $p_1^* > p_2^*$. This implies that we can write $p_1^* = p_2^* + \Delta$, for some $\Delta > 0$. Then

$$2\gamma = \frac{4\hat{x} - 4\left(p_2^* + \Delta\right) + 6p_2^* - 2\hat{x}(p_2^* + \Delta) + 4(p_2^* + \Delta)^2 - 4(p_2^* + \Delta)p_2^* - 2\hat{x}^2 - 2}{4\hat{x} + (p_2^* + \Delta) + p_2^* - \hat{x}\left(p_2^* + \Delta\right) - \hat{x}p_2^* + 2\left(p_2^* + \Delta\right)^2 + 2p_2^{*2} - 4\left(p_2^* + \Delta\right)p_2^* - 2\hat{x}^2 - 2}. \tag{72}$$

This can only be consistent with equilibrium if the numerator is larger than the denominator, that is,

$$-5(p_2^* + \Delta) + 5p_2^* - \hat{x}(p_2^* + \Delta) + 2(p_2^* + \Delta)^2 > -\hat{x}p_2^* + 2p_2^{*2}, \tag{73}$$

or

$$-5\Delta - \hat{x}\Delta + 2(p_2^* + \Delta)^2 > 2p_2^{*2}, \tag{74}$$

hence

$$\Delta(-\hat{x} + 2\Delta + 4p_2^* - 5) > 0. \tag{75}$$

This requires $-\hat{x} + 2\Delta + 4p_2^* - 5 > 0$ thus

$$\Delta > \frac{1}{2}(1 - \hat{x}) + 2(1 - p_2^*). \tag{76}$$

But as $p_2^* < 1/2$, the right-hand side is larger than 1, which is infeasible. Hence we have a contradiction. Thus, we have established that $\gamma > 1/2$ (and thus $a_1^* > a_2^*$) necessarily requires $p_1^* < p_2^*$.

PROPOSITION 5. *With two firms, a uniform distribution of matching values and linear asymmetric advertising technologies, in equilibrium, the more advertising-efficient firm will advertise more.*

Proof. After setting $a_i = a_i^*$, the FOC in advertising levels is

$$0 = p_i^* \frac{a_j^*}{(a_i^* + a_j^*)^2} \left[1 - \hat{x} - p_i^* + p_j^* + \frac{1}{2}(\hat{x}^2 - p_j^{*2}) \right. \\ \left. -(\hat{x} + p_j^* - p_i^*)(1 - \hat{x}) - \frac{1}{2}(\hat{x} - p_i^*)(\hat{x} + 2p_j^* - p_i^*) \right] - \alpha_i . \tag{77}$$

Suppose that firm 1 is the more advertising-efficient firm, so $\alpha_1 < \alpha_2$. From the condition above, we then require

$$p_1^* a_2^* \left[1 - \hat{x} - p_1^* + p_2^* + \frac{1}{2}(\hat{x}^2 - p_2^{*2}) \right. \\ \left. -(\hat{x} + p_2^* - p_1^*)(1 - \hat{x}) - \frac{1}{2}(\hat{x} - p_1^*)(\hat{x} + 2p_2^* - p_1^*) \right] \\ < p_2^* a_1^* \left[1 - \hat{x} - p_2^* + p_1^* + \frac{1}{2}(\hat{x}^2 - p_1^{*2}) \right. \\ \left. -(\hat{x} + p_1^* - p_2^*)(1 - \hat{x}) - \frac{1}{2}(\hat{x} - p_2^*)(\hat{x} + 2p_1^* - p_2^*) \right]. \tag{78}$$

Close inspection of the bracketed terms reveals that they are exactly equal. Hence the inequality simplifies to

$$p_1^* a_2^* < p_2^* a_1^*.$$

Suppose that $a_1^* < a_2^*$. This, by Proposition 4, necessarily implies $p_1^* > p_2^*$, hence $p_1^* a_2^* > p_2^* a_1^*$. But this contradicts the inequality derived above, thus establishing the result.

For completeness, it remains to be shown that the solution to the FOCs is indeed a Nash equilibrium. Proving this analytically is difficult because the system of FOCs cannot be solved in closed form. We proceed by checking it numerically. Let us set $\alpha = 0.75$. Solving numerically the FOCs we obtain that $(a_1^*, p_1^*) = (0.025, 0.479)$ while $(a_2^*, p_2^*) = (0.018, 0.481)$. Taking as given the strategy of player 2, Figure A2 plots the profits of firm 1 when deviating from the equilibrium

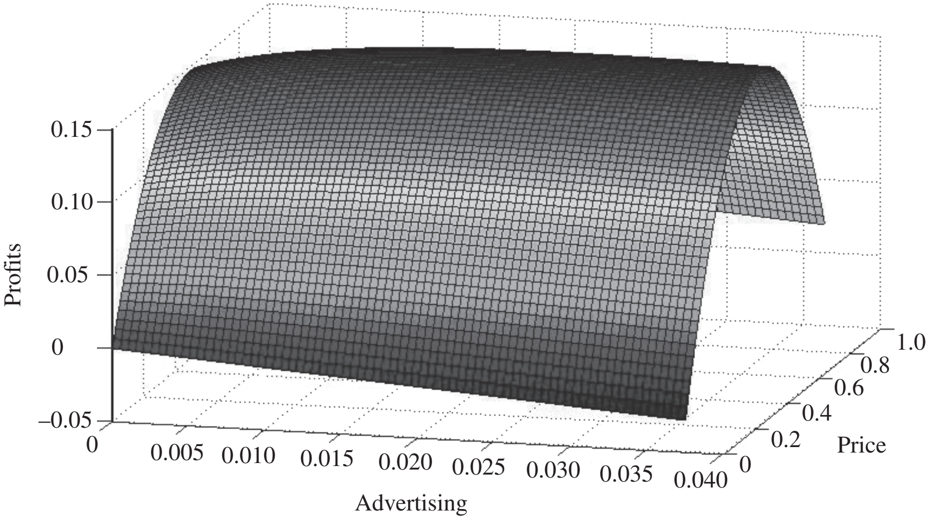

Fig. A2. *Firm 1's Profit Function* ($\mu = 1,\ f = 1,\ \phi_1(a) = 0.75a,\ s = 0.08,\ a_2^* = 0.018,\ p_2^* = 0.481$)

strategy (the graph includes also large deviations, p_1 varies from 0 to 1). The figure clearly shows that the profit function is well-behaved also in this case where the advertising cost function is linear. Setting other values for α does not change the graph qualitatively. As a result, the solution to the FOCs is indeed a Nash equilibrium.

References

Anderson, S.P. and Renault, R. (1999). 'Pricing, product diversity and search costs: a Bertrand–Chamberlin–Diamond model', *Rand Journal of Economics*, vol. 30, pp. 719–35.

Caplin, A. and Nalebuff, B. (1991). 'Aggregation and imperfect competition: on the existence of equilibrium', *Econometrica*, vol. 59(1), pp. 25–59.

Perloff, J. and Salop, S. (1985). 'Equilibrium with product differentiation', *Review of Economic Studies*, vol. 52, pp. 107–20.