Market Microstructure Invariants: Theory and Implications of Calibration *

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Abstract

Using the intuition that financial markets transfer risks in “business time,” we define “market microstructure invariance” as the hypothesis that the size distribution and transaction costs of risk transfers (“bets”) are constant across assets and time. Defining trading activity $W$ as the product of dollar volume and returns standard deviation, invariance predicts that intended order size, market impact costs, and bid-ask spread costs—as fractions of volume and volatility—are proportional to $W^{-2/3}$, $W^{1/3}$, and $W^{-1/3}$, respectively. Using calibration results from structural estimates in a companion empirical paper, we estimate the arrival rate of bets (“market velocity”) and the size distribution of bets, develop formulas for estimating impact and spread costs, and describe two indices of market liquidity.

Keywords: asset pricing, market microstructure, transactions costs, liquidity, invariance, order imbalances, market impact, bid-ask spread.

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1 Introduction

This paper\textsuperscript{1} proposes a modeling principle for financial markets that we call “market microstructure invariance.” Market microstructure invariance begins with the intuition that in financial markets, risk transfer takes place in business time. For actively traded assets, business time passes quickly; for inactively traded assets, business time passes slowly. Market microstructure invariance hypothesizes that when trading in financial assets is scaled in units of business time instead of calendar time, the rate at which risks are transferred, the sizes of the risks transferred, the market impact costs of transferring risks, and the bid-ask spread costs of transferring risks will be the same for assets with different levels of calendar-time trading activity.

The idea of using invariance principles in finance and economics, at least implicitly, is not new. For example, Modigliani-Miller theory (1958) is an example of an invariance principle. The idea of measuring trading in financial markets in business time or transaction time is not new either. The “time-change” literature has a long history, beginning with Mandelbrot and Taylor (1967), who link business time to transactions, and Clark (1973), who links business time to volume. More recent papers include Hasbrouck (1999), Ané and Geman (2000), Dufour and Engle (2000), Plerou et al. (2000), and Derman (2002).

Microstructure invariance shifts the intuition of this literature from understanding the relationship between trading volume and business time to understanding the relationship between risk transfer and business time. To emphasize the concept of risk transfer, we measure trading activity as the product of dollar volume and volatility. Our invariance hypotheses are based on the intuition that both volume and volatility should be understood in units of business time, not just volume alone. As a result of expressing both dollar trading volumes and returns volatility in units of business time, we show that apparent differences in the trading characteristics of different assets turn into invariance relationships.

When portfolio managers trade financial assets, they can be modeled as playing trading games in business time. We refer to innovations in the trading intentions of these investors as “bets.” Since trading in financial assets exchanges risks and their sizes depend on both notional value and returns volatility, we measure the size of bets as the product of dollar size and volatility. We make an important distinction between bets and actual “order flow” in the market. The order flow is the sequence of buy and sell orders which implement bets. Since the execution of one bet may lead to numerous orders generating many trades spread out over time, there may be many autocorrelated trades implementing bets. This order flow is intermediated by non-bet trading volume from market makers, high frequency traders, statistical arbitragers, and other short-term trading strategies. This makes calculating the variance of order

\textsuperscript{1}We have divided the long manuscript “Market Microstructure Invariants” (May, 2011) into two separate papers: this paper and our companion empirical paper “Market Microstructure Invariants: Empirical Evidence from Portfolio Transitions” (December 2011).
flow imbalances from actual order flow difficult. In contrast, since bets are defined as approximately independently distributed, the variance of order flow imbalances can be calculated as the sum of the variances of the individual bets.

We define the “business-time” clock so that one new bet arrives, on average, for each unit of business time that passes. “Market microstructure invariance” is a combination of the following three distinct invariance conjectures concerning the way in which trading games are similar:

- “Trading game invariance” hypothesizes that over each unit of business time, the distribution of the dollar mark-to-market gains and losses of bets is the same across assets and across time.
- “Market impact invariance” hypothesizes that the expected dollar market impact cost of a bet is constant across assets and across time.
- “Bid-ask spread invariance” hypothesizes that the expected dollar bid-ask spread cost of a bet is constant across assets and across time.

To construct a measure of aggregate daily risk transfer, we define “trading activity,” denoted $\mathcal{W}$, as the product of daily dollar volume and daily percentage standard deviation of returns. Given an empirical proxy for bets, the three invariance conjectures lead to the following empirical predictions about how the size of bets, market impact costs of bets, and bid-ask spread costs of bets vary with the level of trading activity:

- Trading game invariance predicts that the arrival rate of bets is proportional to $\mathcal{W}^{2/3}$ and, equivalently, that the size of bets is proportional to $\mathcal{W}^{1/3}$. If $\tilde{Q}$ denotes the random number of shares in a bet and $V$ denotes expected daily trading volume in shares, then the distribution of the random variable $\mathcal{W}^{2/3} \cdot \frac{\tilde{Q}}{V}$ is invariant, in the sense that it does not vary across assets and time.
- Market impact invariance predicts the expected market impact cost (per dollar traded in volatility units) incurred by executing a bet equal to a given fraction of average daily volume (say one percent) is proportional to $\mathcal{W}^{1/3}$.
- Bid-ask spread invariance predicts that the expected bid-ask spread cost (per dollar traded in volatility units) incurred by executing a bet is proportional to $\mathcal{W}^{-1/3}$.

These predictions are based on simple intuition. Imagine speeding up a trading-game clock. This increases proportionally both the number of bets per day and the variance of risks transferred by each bet. Since the risk transferred by a bet is measured based on standard deviation and standard deviation is the square root of variance, the risk transferred by a bet increases only half as fast as the number of bets per day. Trading activity is the product of the number of bets per day and
the average bet size. As trading activity increases as a result of a faster trading-game clock, trading game invariance implies that 2/3 of the increase represents the increased arrival rate of bets and 1/3 of the increase represents the increased risk transferred by each bet, per unit of calendar time.

This simple intuition requires careful interpretation. The invariance hypothesis does not imply that an increase in trading activity increases the volatility half as fast as dollar trading volume, per unit of calendar time. For example, speeding up the trading-game clock can increase the arrival rate of bets twice as fast as the dollar size of bets, leaving calendar-time volatility unchanged.

We describe two alternative invariance principles. We call these alternatives the model of “invariant bet frequency” and the model of “invariant bet size.” The model of invariant bet frequency, which is similar to Amihud (2002) and captures the conventional wisdom of traders, is based on the assumption that as trading activity increases, the number of bets per calendar day remains constant, but their size scales up proportionally. The model of invariant bet size, which captures some of the intuition of Hasbrouck (2009) and Gabaix et al. (2006), is based on the assumption that as trading activity increases, the size of bets remains constant, but the number of bets per calendar day increases. In comparison with our proposed model of market microstructure invariance, these alternative invariance principles make dramatically different predictions about how bet size, market impact costs, and bid-ask spread costs vary with the level of trading activity. All three models nest conveniently into common specifications amenable to structural estimation. Determining which model better describes the data is ultimately an empirical question.

**Microstructure Invariance and Microstructure Literature.** Microstructure invariance builds a bridge from theoretical models of market microstructure to empirical tests of those models.

Many different theoretical models use game theory to model securities trading. These models typically make specific assumptions about the consistency of beliefs across traders, the flow of public and private information which informed traders use to trade, the flow of orders from liquidity traders, and auction mechanisms in the context of which market makers compete to take the other side of trades. Some models emphasize adverse selection, such as Treynor (1971), Kyle (1985), Glosten and Milgrom (1985), and Back and Baruch (2004); some models emphasize inventory dynamics, such Grossman and Miller (1988) and Campbell and Kyle (1993); and some models emphasize both, such as Grossman and Stiglitz (1980) and Wang (1993).

There is a voluminous empirical market microstructure literature describing how the rate at which orders arrive in calendar time, the dollar size of orders, the market impact costs, and bid-ask spread costs vary across different assets. Hasbrouck (2007) provides a good description of this literature. For example, Brennan and Subramanyam (1998) estimate order size as a function of various stock characteristics.

Theoretical microstructure models suggest that order flow imbalances move prices.
Particular parameters describing how order flow moves prices depend on the specifics of each model. Understanding the cross-sectional empirical implications concerning how order flow moves price has been difficult. The theoretical microstructure models do not provide a unified framework for mapping the theoretical concept of an order flow imbalance into its empirical measurement for stocks with different levels of trading activity. Instead, researchers have used imperfect empirical proxies for order flow imbalances, such as the difference between uptick and downtick volume, an approach popularized by Lee and Ready (1991). Even when order flow imbalances can be measured, the theoretical models often do not provide precise predictions concerning how price impact varies across different stocks. As a result, some researchers have taken a purely empirical approach, such as regressing price changes on Lee and Ready (1991) imbalances to obtain market impact coefficient and then relating it to stock characteristics such as market capitalization, trading volume, and volatility. Breen, Hodrick, and Korajczyk (2002) is an example of this approach.

By contrast, microstructure invariance generates precise and empirically testable predictions about how the size and arrival rates of intended orders, as well as market impact and bid-ask spread costs, vary across assets with different levels of trading activity. These predictions are based on a common intuition shared by many models. In this sense, market microstructure invariance is a modeling principle applicable to different models, not a model itself. The usually unidentifiable parameters of various theoretical models show up in invariance constants, which can be calibrated from the data.

Consider, for example, the continuous time theoretical model of Kyle (1985). The price impact of one share of stock $\lambda$ is shown to be given by $\lambda = \sigma_V/\sigma_U$, where $\sigma_V$ is fundamental volatility and $\sigma_U$ is the standard deviation of order flow imbalances. The value of $\sigma_V$ is easily inferred from a stock’s price and returns volatility, under the maintained hypothesis that risk neutral market makers make markets semi-strong form efficient. The value of $\sigma_U$ is more difficult to measure empirically. Intuitively, $\sigma_U$ should be related to trading volume in some way. The continuous model provides no help concerning what this relationship is; indeed, the Brownian motion assumption for order flow implies that trading volume should be infinite. Without some approach for measuring $\sigma_U$, the model is untestable.

An approach consistent with the spirit of the model is to approximate the order flow process with a compound Poisson process, in which case $\sigma_U$ is a function of the arrival rate and size of orders. This approach is also consistent with the spirit of other models, such as Glosten and Milgrom (1985). Amihud (2002) then effectively makes the invariance assumption that the arrival rate is some unknown constant, the same for all stocks, as a result of which $\sigma_U$ is proportional to trading volume. Hasbrouck (2009) effectively makes the invariance assumption that the dollar size distribution of the orders is some unknown constant, the same for all stocks, as a result of which $\sigma_U$ is proportional to the square root of trading volume. Market microstructure invariance makes the assumption that the risk transferred by an order per unit of business time
is constant. This leads to a structural relationship in which the product of dollar volume and volatility implies the number and sizes of independent orders (“bets”) arriving in the market. This makes it possible to calculate the standard deviation of order flow imbalances $\sigma_U$ from data on dollar trading volume and volatility and then formulate precisely the testable predictions based on the formula $\lambda = \sigma_V/\sigma_U$ from Kyle (1985) concerning how market depth relates to volume and volatility.

The Power of Invariance Principles. In other areas of science, invariance principles are powerful. For example, a fundamental Galilean invariance principle of Newtonian physics is that the laws of physics do not vary with the speed at which an object is traveling. This invariance principle simplifies the laws of physics. Physicists need only deal with a small number of the physical laws based on a handful of physical “constants,” instead of a potentially infinite set of complicated laws based on numerous constants describing objects traveling at different speeds.

Applying this same principle to market microstructure, we think of the “velocity” of the market for a particular asset as the speed with which business time passes for that asset, when measured in units of calendar time. Market microstructure invariance is the principle that underlying “constants” relating to the size of orders, market impact costs, and bid-ask spread costs do not vary across markets with different velocities.

In both physics and market microstructure, application of invariance principles requires that certain assumptions be met. For example, the laws of physics hold in simplest form for objects traveling in a vacuum. These laws are modified when resistance from air generates friction. Similarly, in market microstructure, we believe that the invariance relationships hold under idealized conditions. For example, our invariance relationships may assume an idealized environment with features like very small tick size, reasonably competitive market makers, and minimal transactions fees and taxes. The invariance relationships may need modification if, for example, a large tick size, non-competitive market access, or high fees and taxes increase bid-ask spreads substantially. Invariance principles provide a benchmark from which the importance of such frictions can be measured.

The potential benefits of invariance principles for empirical market microstructure are enormous. In the area of transactions costs measurement, for example, controlled experiments are costly and natural experiments are rare. Furthermore, even well-specified tests of transactions costs models tend to have low statistical power. Market microstructure invariance defines parsimonious structural relationships leading to precise predictions about how order size and transactions costs vary across stocks with different dollar volume and volatility. These predictions can be tested with structural estimates of a handful of parameters using limited data from many different stocks. Below we use results from our companion empirical paper to calibrate the values of these common constants.
Results from Companion Empirical Paper. Our companion empirical paper Kyle and Obizhaeva (2011) tests the predictions of market microstructure invariance and the two alternative models using a database of more than 400,000 portfolio transition orders executed over the period 2001-2005 by a leading vendor of portfolio transition services. Portfolio transitions are economically significant transactions initiated by institutional sponsors who transfer funds from a legacy portfolio manager to a new manager to replace fund managers, change asset allocation, or accommodate cash inflows and outflows. Portfolio transitions provide a good natural experiment for measuring transactions costs. Microstructure invariance predicts that order size, market impact costs, and bid-ask spread costs—as fractions of daily volume and volatility—are proportional to \( W^{2/3}, W^{1/3}, \) and \( W^{-1/3} \), respectively. We make the identifying assumption that portfolio transition orders are proportional to bets. While not in perfect agreement with the predictions, the estimated exponents of -0.63, 0.33, and -0.39 are remarkably close to the predicted values of \(-2/3, 1/3, \) and \(-1/3, \) respectively.

By imposing the predicted values of these three exponents, the companion paper calibrates values of four parameters characterizing invariant constants. These calibrated values are based on the two identifying assumptions that (1) portfolio transition orders are the same size as bets (a tighter assumption than proportionality above) and (2) bet volume is 50% of total volume (as would be the case if a specialist intermediated all bets). The calibrated values are expressed in terms of a “benchmark stock” with price of $40 per share, trading volume of one million shares per day, and volatility of 2% per day, i.e. \( W = 40 \cdot 10^6 \cdot 0.02 \).

Two of the four calibrated parameter values describe the size distribution of bets. For the benchmark stock, the mean order size is estimated to be 0.34% of expected daily volume (85 bets per day). Conditional of the level of trading activity, portfolio transition orders for all stocks are found to have a distribution symmetric about zero, with unsigned order size conforming closely to a log-normal distribution with log-variance of 2.50.

Two of the calibrated parameter values describe impact and spread costs of executing bets in the benchmark stock. The market impact cost of trading one percent of expected daily volume is 2.89 basis points, and the half bid-ask spread cost is 7.90 basis points.

Implications of Calibrated Parameter Values. As we describe in this paper, the invariance hypothesis and calibration results from our companion paper imply simple formulas describing how markets for individual assets work. Bets arrive approximately randomly at a rate proportional to \( W^{2/3} \) (see equation(32)), and their size is distributed closely to a log-normal (see equation (31)). Letting \( P \cdot V \) denote expected dollar volume and \( \sigma \) expected daily returns standard deviation, the invariance hypothesis implies simple and easily implementable formula for transactions costs, in which the impact costs of trading a given percent of expected daily volume are propor-
tional to \((P \cdot V)^{1/3} \cdot \sigma^{4/3}\) and bid-ask spread costs are proportional to \((P \cdot V)^{-1/3} \cdot \sigma^{2/3}\) (see equations (30) and (34)). All these formulas take as their only input expected prices, daily volume, and daily percentage returns volatility. These formulas extrapolate the parameters, calibrated for the benchmark stocks, to assets with different levels of trading activity, by scaling calibrated parameters up and down, using the elasticities implied by microstructure invariance.

We find that for all stocks, the invariant expected market impact costs and spread costs are $1960 and $373 per bet, respectively. The expected market impact cost and spread cost of the median bet are only $13.20 and $107, respectively. These calibration results are consistent with conventional traders’ wisdom that market impact costs dominate spread costs for large trades, whereas spread costs dominate market impact costs for small trades.

The fact that bet size is log-normally distributed with log-variance of 2.50 implies that large bets dominate small bets in terms of their contribution to trading volume and return variance. For example, a log-variance of 2.50 implies that the larger half of bets account for 94.29% of bet trading volume and generates 99.92% of total returns variance resulting from linear price impact.

We derive two liquidity indices for the markets for individual stocks (see equations (36) and (37) below). The first index measures the expected cost of converting an asset to cash as a fraction of the dollar value of the asset (in basis points). This liquidity measure is inversely proportional to the cube root of dollar volume per unit of returns variance, i.e., \((P \cdot V/\sigma^2)^{-1/3}\). The second index is defined as the first index divided by returns volatility. Analogously to a Sharpe ratio, it measures in units of returns standard deviation the expected cost of transferring a risk of the sizes that are exchanged in the market. This measure is proportional to \((P \cdot V \cdot \sigma)^{-1/3}\), or the square root of “market velocity,” measured by the number of bets per day or the speed with which business time passes. For the benchmark stock, the calibrated values of the liquidity measures are 50 basis points and 0.25 daily standard deviation units, respectively.

Plan of this Paper. The remainder of this paper, section 2 develops market microstructure invariance, section 3 suggests two alternatives hypothesis, and section 4 presents implications of calibration using empirical estimates from our companion empirical paper.

2 Market Microstructure Invariance

Microstructure characteristics such as order size, order arrival rate, price impact, and bid-ask spreads vary across assets and across time. We refer to “market microstructure invariance” as the hypothesis that this variation almost disappears, if the market microstructure of the trading processes is examined at a business time scale, which
measures the rate at which risk transfer takes place.

In order to examine trading at different time scales, we avoid making explicit assumptions about the structure of information, the motivations of traders, and the consistency of their beliefs. To develop a dynamic equilibrium microstructure model, we would have to address complicated modeling issues. Instead, we take a reduced form approach, focusing on the effects of different time scales.

2.1 Bets and Trading Volume

We think of trading a stock as playing a trading game, in which long-term traders buy and sell shares to implement “bets” in the market. Intermediaries with short-term trading strategies—market makers, high frequency traders, and other arbitragers—clear markets by taking the other side of bets placed by long-term traders. Observed trading volume is thus the sum of bet volume by long-term traders and non-bet volume by short-term intermediaries.

A bet represents an innovation in the intended order flow from long-term investors. Consider an asset manager engaging in a research process to identify over-valued and under-valued stocks. The result of the research process might be a decision to purchase 100,000 shares of IBM stock. This decision is a bet. As innovations, bets are independently distributed. The independence of bets will be very important later, when we start thinking about order imbalances and their relation to price volatility.

Bets are difficult for researchers to observe. Consider the asset manager who has decided to purchase 100,000 shares of IBM stock. The trader might implement this bet by placing a sequence of orders to purchase 20,000 shares of stock per day for five days in a row. Each of these orders might be broken into smaller pieces for execution. For example, on day one there may be trades of 2,000, 3,000, 5,000, and 10,000 shares executed at different prices. Each of these smaller trades may show up in TAQ data as multiple “prints.” Since the various orders, trades, and prints implementing bets are inherently positively correlated, it would not be appropriate to think of them as separate statistically independent bets. Bets represent independent increments in the intended order flow, not potentially correlated increments in the actual order flow.

A bet is like a new idea, which can be shared. If an analyst recommendation to buy a stock is followed by buy orders from multiple customers, all of these orders can be considered part of the same bet. For example, if an analyst recommends to ten different customers to buy a stock, and all of the customers together buy 100,000 shares as a result of the recommendation, it is reasonable to think of the 100,000-share purchase as one bet, rather than as a sequence of ten different bets. Since the ten purchases are all based on the same information—the recommendation of the analyst—these decisions probably lack statistical independence.

As a first approximation, we assume that “bets” are generated by a Poisson process. Bets arrive approximately randomly with an arrival rate of \( \gamma \) bets per day. Bets are approximately independently distributed, with positive quantities representing
buying and negative quantities representing selling. Let $\tilde{Q}$ denote a random variable whose distribution reflects the size of bets. We assume $\tilde{Q}$ symmetric about zero with $E\{\tilde{Q}\} = 0$. To simplify notation, we suppress subscripts denoting time and stock identifiers.

We do not expect the bet generating process to conform precisely to the definition of a compound Poisson process, which requires constant bet arrival rates with identically and independently bet sizes. Instead, over short periods of time, we expect some variation in bet arrival rates, while over long periods of time, we expect a small negative serial correlation in bets, so that markets might clear “in the long run” without the presence of intermediaries. Moreover, bet arrival rates and sizes of bets may vary over time as the trading characteristics of a stock change. For example, if large price increases change a stock from a small stock with low level of trading activity to a big stock with a high level of trading activity, we might expect larger and more frequent bets to be placed. We think about the assumption of a compound Poisson process generating bets as being a first-order approximation, valid over limited periods of time, such as one month.

Let $V$ denote expected daily trading volume for a stock, measured in shares per day. Since each unit of share volume has a buy-side and a sell-side, the expected share volume from matching buy bets and sell bets is given by $(1/2) \cdot \gamma \cdot E\{|\tilde{Q}|\}$. Expected total daily share volume $V$ is a multiple of this quantity. Letting $\zeta$ denote the volume multiplier, expected daily volume is given by

$$V = \frac{\zeta}{2} \cdot \gamma \cdot E\{|\tilde{Q}|\}. \quad (1)$$

The volume multiplier $\zeta$ reflects the amount of intermediation in the market. If all trades were bets and there are no intermediaries, then we would have $\zeta = 1$, since each unit of trading volume would match a buy-bet with a sell-bet. To the extent that buy and sell bets do not match against each other perfectly, because intermediaries like market makers and arbitragers are involved in the trading process, we would have $\zeta > 1$. For example, if the NYSE specialist were to intermediate all bets without involvement of other intermediaries, we would have $\zeta = 2$. If each bet were intermediated by different NASDAQ market makers, who each has to lay off inventories by trading with another intermediary, then we would have $\zeta = 3$. If positions are passed around among multiple intermediaries, we would have $\zeta = 4$ or more. Calibration of $\zeta$ is an interesting empirical question.

### 2.2 Bet Size and Trading Activity

Since financial markets are places where risks are transferred, a good measure of the size of a bet is the size of the risk transferred. Let $P$ denote the price of the stock in dollars per share. Let $\sigma$ denote the percentage standard deviation of returns on the stock per day (“volatility”). We measure the risk transferred by a bet per calendar
day, denoted $\tilde{B}$, as the product of the notional value $\tilde{Q} \cdot P$ and the daily volatility $\sigma$:

$$
\tilde{B} = \tilde{Q} \cdot P \cdot \sigma.
$$

When comparing transactions across different risky assets, it makes sense to look at the size of the risk transferred, rather than the dollar size of a transaction. For bonds, for example, a given dollar value of volatile long-duration Treasury bonds is much riskier than the same dollar value of short-duration Treasury bills. Thus, our measure of risk transfer includes both the notional value of the bet and the risk per unit of notional value.

Consistently with the idea that the risk of one transaction is measured by the product of dollar value and volatility, we define “trading activity,” denoted $W$, for a risky asset as the product of the daily dollar trading volume and the daily returns standard deviation of the asset (“volatility”):

$$
W = V \cdot P \cdot \sigma.
$$

Trading activity measures the standard deviation of one day’s close-to-close dollar gains or losses resulting from the purchase or sale of one day’s trading volume. In this sense, trading activity $W$ measures the aggregate risk transfer expected to take place in one calendar day.

Since trading volume, stock price, and volatility can all be observed or estimated empirically, trading activity is empirically observable. Observable trading activity can be equivalently defined as the product of three quantities which are difficult to observe: one-half the volume multiplier $\zeta$, the expected number of bets per day $\gamma$, and expected (unsigned) bet size $E\{|\tilde{B}|\}$:

$$
W = \frac{\zeta}{2} \cdot \gamma \cdot E\{|\tilde{B}|\}.
$$

To interpret the empirical results in this paper, we make the identifying assumption that the bet volume multiplier $\zeta$ is a constant, which is the same for all stocks. Investigating whether this assumption holds in the data is an interesting subject for future research. For example, it is reasonable to investigate whether high frequency trading is a larger percentage of trading volume for stocks with high levels of trading activity than for stocks with low levels of trading activity. Under the identifying assumption of constant $\zeta$, it is clear from equation (4) that trading activity $W$ can increase because either the arrival rate of bets $\gamma$ increases or expected bet size $E\{|\tilde{B}|\}$ increases, or both.

Our definitions of trading activity and bets are consistent with the Modigliani-Miller irrelevance of leverage. For example, if a company levers up its equity by paying a debt-financed cash dividend equal to fifty percent of the value of the equity, then the volatility of the remaining equity, ex dividend, doubles, while the price halves. If share volume remains constant, then dollar volume halves, but trading activity and
bet size remain unaffected by the change in leverage. This is consistent with the intuition that each share of leveraged stock still represents the same pro rata share of firm risk as a share of un-leveraged stock.

Our definitions of trading activity and bets are also consistent with the Modigliani-Miller irrelevance of stock splits. For example, a two-for-one stock split should theoretically double the share volume of bets without affecting volatility. Since each share is worth one-half of its pre-split value, then our measures of bets and trading activity are not affected by a two-for-one stock split, which only doubles the share size of bets.

2.3 Three Principles of Microstructure Invariance

The invariance hypothesis captures the intuition that trading games for securities with different levels of trading activity are the same, except for being played at different speeds. Like the game of chess, the game of trading financial assets can be played quickly or slowly. Trading an active stock is like playing chess with a fast time clock. Trading an inactive stock is like playing the same game with a slow time clock.

To develop the implications of microstructure invariance, we re-scale time so that one bet is expected to arrive in the market for each tick of the business-time clock. Note that since the concept of a bet is different from a concept of a transaction, the business-time clock implied by our model is different from the transaction-time clock of Mandelbrot and Taylor (1967). Our main intuition is that re-scaling time not only creates time units in which bets arrive at the same rate, but also the size of bets and the costs of their execution are the same as well.

Using this idea, we make three invariance conjectures concerning (1) the size of bets, (2) the price impact costs of bets, and (3) the bid-ask spread costs of bets.

Trading Game Invariance: Between each tick on the re-scaled clock, the distribution of the risks transferred by a bet does not vary across assets and across time. If \( \gamma \) bets are expected to arrive per day, then the business time clock runs faster than the calendar time clock by a factor of \( \gamma \). The quantity \( \frac{\tilde{B}}{\gamma^{1/2}} \) measures the signed standard deviation of the dollars gained or lost on a bet over a time interval in business time, with positive values for buys and negative values for sells. Letting \( \tilde{I} \) denote a random variable with this “invariant” distribution, trading game invariance can be stated as follows:

\[
\tilde{I} \approx \frac{\tilde{B}}{\gamma^{1/2}} = \frac{P \cdot \tilde{Q} \cdot \sigma}{\gamma^{1/2}},
\]

where the notation “\( \approx \)” is interpreted to mean “has the same probability distribution as.” Trading game invariance says that the probability distribution for \( \tilde{I} \) is invariant across assets and across time, even though the distributions of \( \tilde{B} \) and \( \tilde{Q} \) as well as the values of \( P, V, \sigma, \) and \( \gamma \) vary across assets and across time.
This invariance is based on the idea that trading games differ only in the speed with which they are being played. Speeding up the time clock speeds up proportionately both bet arrival rate and bet variance per calendar day. Since bet risk is based on standard deviation, not variance, taking a square root to convert variance to standard deviation implies that, as trading activity increases, bet risk $|\tilde{B}|$ increases half as fast as the bet arrival rate $\gamma$. This implies the invariance of $\tilde{I}$ across trading games with different business-time clocks. Our definition of $\tilde{I}$ is therefore consistent with the irrelevance of the speed of the time clock. The invariance of the distribution of $\tilde{I}$ is also unaffected by stock splits and changes in leverage.

Note that the probability distributions of $\tilde{I}$, $\tilde{B}$, and $\tilde{Q}$ all have the same shape. The distributions of $\tilde{B}$ and $\tilde{Q}$ differ from $\tilde{I}$ only by proportionality coefficients $\gamma^{-1/2}$ and $P \cdot \sigma \cdot \gamma^{-1/2}$, respectively. For example, this hypothesis implies that as the bet arrival rate $\gamma$ increases, the distribution of unsigned bet sizes $|\tilde{B}|$ shifts upwards half as fast as the bet arrival rate changes.

Using equations (4) and (5), we can express both the bet arrival rate $\gamma$ and bet size $\tilde{B}$ as functions of trading activity $W$:

$$\gamma = \left[ \frac{\zeta}{2} \cdot E\{|\tilde{I}|\} \right]^{-2/3} \cdot W^{2/3}, \quad (6)$$

$$\tilde{B} \approx \left[ \frac{\zeta}{2} \cdot E\{|\tilde{I}|\} \right]^{-1/3} \cdot W^{1/3} \cdot \tilde{I} \quad (7)$$

In these equations, $W$ varies across assets and across time, but the coefficients multiplying $W$—powers of $(\zeta/2) \cdot E\{|\tilde{I}|\}$—are constants. These equations imply that if $W$ increases by one percent, the arrival rate of bets $\gamma$ increases by 2/3 of one percent and the distribution of bet size $\tilde{B}$ shifts upwards by 1/3 of one percent.

In practice, the size of an institutional bet is often expressed as a fraction of average daily volume, which in our notation is $\tilde{Q}/V$. Since $Q/V = \tilde{B}/W$, expressing $\tilde{Q}/V$ as a function of trading activity yields

$$\frac{\tilde{Q}}{V} \approx \left[ \frac{\zeta}{2} \cdot E\{|\tilde{I}|\} \right]^{-1/3} \cdot W^{-2/3} \cdot \tilde{I}. \quad (8)$$

Note that in equations (6), (7), and (8), parameter $\zeta$ effectively deflates both trading volume $V$ and trading activity $W$ by eliminating non-bet volume from these variables.

**Market Impact Invariance:** The expected market impact cost of a bet is the same across assets and across time. Let $\lambda$ denote the linear market impact cost of buying or selling one share of stock, measured in units of dollars per share per share, or “dollars per share-squared.” This notation for $\lambda$ is consistent with Kyle (1985). Assuming linear price impact costs, the expected impact cost of a bet is a quadratic function of bet size given by $(\lambda/2) \cdot E\{\tilde{Q}^2\}$, with $\lambda$ and $\tilde{Q}$ varying across assets and across time. Let $C_L$ denote a constant dollar impact cost which does
not vary across stocks or across time. Then market impact cost invariance can be expressed as

$$C_L = \frac{\lambda}{2} \cdot E\{\hat{Q}^2\}, \quad (9)$$

where $\lambda$ and $\hat{Q}$ vary across assets and across time, but $C_L$ does not. Note that $\lambda \cdot E\{\hat{Q}^2\}$ in equation (9) is multiplied by $1/2$, reflecting the implicit identifying assumption that traders execute bets by walking up the demand curve. If other market participants can detect large bets during execution, then the parameter $1/2$ can potentially have a larger value.

Using equation (9), the percentage market impact of trading $\hat{Q}$ shares is given by,

$$\frac{\lambda}{P} \cdot |\hat{Q}| = 2 \cdot C_L \cdot \left[\frac{\zeta}{2}\right]^{2/3} \cdot \left[\frac{E\{\hat{I}\}}{E\{\hat{I}^2\}}\right]^{2/3} \cdot W^{1/3} \cdot \sigma \cdot |\hat{Q}| \cdot V. \quad (10)$$

Under the combined assumptions of trading game invariance and market impact invariance, the variables $\hat{I}$ and $C_L$ in equation (10) do not vary across assets and across time. This equation leads to the empirical hypothesis that the expected market-impact cost of trading a fixed percentage of daily volume accounts for a fraction of daily volatility proportional to $W^{1/3}$, with a proportionality constant $2 \cdot C_L \cdot \left[\zeta/2\right]^{2/3} \cdot [E\{\hat{I}\}]^{2/3} \cdot E\{\hat{I}^2\}]^{-2/3}$ which does not vary across assets and across time (assuming $\zeta$ is also constant).

**Bid-Ask Spread Invariance:** The expected bid-ask spread cost of a bet is the same across assets and across time. Let $\kappa$ denote the bid-ask spread, measured in dollars per share. The expected bid-ask half-spread cost of a bet, expressed in dollars, is given by $(\kappa/2) \cdot E\{|\hat{Q}|\}$. Let $C_K$ denote the invariant dollar half-spread cost of a bet, which does not vary across assets and across time. Then bid-ask spread invariance implies

$$C_K = \frac{\kappa}{2} \cdot E\{|\hat{Q}|\}, \quad (11)$$

where $\kappa$ and $\hat{Q}$ vary across stocks and across time, but $C_K$ does not.

Using equation (11), the percentage expected bid-ask spread cost of a one-share round-trip trade is given by,

$$\frac{\kappa}{P} = 2 \cdot C_K \cdot \left[\frac{\zeta}{2}\right]^{1/3} \cdot [E\{\hat{I}\}]^{-2/3} \cdot W^{-1/3} \cdot \sigma. \quad (12)$$

This equation leads to the empirical hypothesis that the expected bid-ask spread cost of trading a share accounts for a fraction of daily volatility proportional to $W^{-1/3}$, where the proportionality constant, given by $2 \cdot C_K \cdot \left[\zeta/2\right]^{1/3} \cdot [E\{\hat{I}\}]^{-2/3}$, does not vary across assets and across time (assuming $\zeta$ constant).

Market microstructure invariance is a combination of the three hypotheses of (1) trading game invariance, (2) market impact invariance, and (2) bid-ask spread invariance.
3 Alternative Hypotheses

We examine two alternatives to market microstructure invariance. These alternative hypotheses serve three purposes. First, they provide empirical alternatives which can be easily tested against our proposed model, because they nest into a common specification. Second, the two alternatives are similar in spirit to the models of Amihud (2002) and Hasbrouck (2009), respectively. In this sense, the alternative models capture the thinking behind the current literature. Third, we believe that the alternative models capture the intuition traders use to think about trading costs.

Alternative Models and Bet Size  To specify alternative hypotheses we generalize the invariance equation (5). Let $\alpha$ denote a parameter such that $0 \leq \alpha \leq 1$. The generalized invariance relationship is

$$\tilde{I} \approx \frac{\tilde{B}}{\gamma^\alpha/(1-\alpha)} = \frac{P \cdot \tilde{Q} \cdot \sigma}{\gamma^\alpha/(1-\alpha)}.$$  \hspace{1cm} (13)

For $\alpha = 1$, we interpret this equation to mean that $\gamma$ is a constant, and there is no well-defined invariant distribution $\tilde{I}$.\footnote{Assuming $\tilde{B}$ is symmetric around zero, equation (13) could have been written $\tilde{J} = (1-\alpha) \ln(|\tilde{B}|) - \alpha \ln(\gamma)$, with $\tilde{J}$ being a new log version of the invariant distribution consistent with $\tilde{J} = (1-\alpha) \ln(|\tilde{I}|)$ for $\alpha \neq 1$. When $\alpha = 1$, $\tilde{J}$ is well-defined with $\tilde{J} = -\ln(\gamma)$ even though $\tilde{I}$ is not defined.} Equation (5) of market microstructure invariance corresponds to $\alpha = 1/3$. Generalized versions of equations (6), (7), and (8) can be easily derived by solving equations (4) and (13) for $\gamma$, $\tilde{B}$, and $\tilde{Q}/V$ in terms of $\zeta$, $W$, and moments of $\tilde{I}$.

Let $\tilde{Q}^*$, $V^*$, $\zeta^*$, and $W^*$ denote order size, daily expected share volume, the volume deflator, and daily trading activity for a hypothetical benchmark stock which conforms to equations (2) through (4). Dividing the generalizations of equations (6), (7), and (8) for some stock by corresponding equations for the benchmark results in cancelation of moments of $\tilde{I}$ and yields the following equations similar to equations (6), (7), and (8):

$$\gamma = \gamma^* \cdot \left[ \frac{\zeta}{\zeta^*} \right]^{-(1-\alpha)} \cdot \left[ \frac{W}{W^*} \right]^{1-\alpha},$$  \hspace{1cm} (14)

$$\tilde{B} \approx \tilde{B}^* \cdot \left[ \frac{\zeta}{\zeta^*} \right]^{-\alpha} \cdot \left[ \frac{W}{W^*} \right]^{\alpha},$$  \hspace{1cm} (15)

$$\frac{\tilde{Q}}{V} \approx \frac{\tilde{Q}^*}{V^*} \cdot \left[ \frac{\zeta}{\zeta^*} \right]^{-\alpha} \cdot \left[ \frac{W}{W^*} \right]^{-(1-\alpha)}.$$  \hspace{1cm} (16)

The ratio $\zeta/\zeta^*$ here adjusts both trading volume $V$ and trading activity $W$ for non-bet volume. The three equations (14), (15), and (16) are all equivalent ways of expressing the same empirical predictions. Assuming $\zeta = \zeta^*$ for all stocks, these equations state that as trading activity increases, a fraction $1-\alpha$ of the increase results...
from increased bet arrival rate and therefore a fraction \( \alpha \) of the increase results from increased bet size, implying that bet size as a fraction of trading activity declines at a rate \( 1 - \alpha \). Market microstructure invariance is equivalent to the hypothesis

\[
\alpha = 1/3,
\]

implying bet frequency increases twice as fast as bet arrival rate, as trading activity increases.

Each value of \( \alpha \) in the range \( 0 \leq \alpha \leq 1 \) corresponds to a different alternative model. We focus on two alternative models, corresponding to \( \alpha = 1 \) and \( \alpha = 0 \).

Our first alternative model, which we call “the model of invariant bet frequency,” is based on the hypothesis that the number of bets per day is invariant across assets with different levels of trading activity. This model implies that as trading activity increases, bet size increases proportionately, but the expected number of bets per day is constant. Thus, bet size as a fraction of volume is constant across assets and across time. The model of invariant bet frequency is equivalent to the hypothesis

\[
\alpha = 1.
\]

Our second alternative model, which we call “the model of invariant bet size,” is based on the hypothesis that the size of bets is invariant across assets with different levels of trading activity. This model implies that as trading activity increases, the number of bets increases proportionately, but bet size does not change. The model of invariant bet size is equivalent to the hypothesis

\[
\alpha = 0.
\]

Figure 1 illustrates differences between market microstructure invariance and the two alternatives by comparing two hypothetical stocks. One stock has four equal-size bets per day \( (\gamma^* = 4) \). Its expected daily volume is 1 million shares \( (V^* = 10^6) \). Thus, each bet of 250,000 shares \( (Q^* = 250,000) \) contributes one-fourth of average daily volume. The other stock that has the same volatility, price, and volume deflator, but daily volume of 8 million shares \( (V = 8 \cdot 10^6) \).

The three models differ in how they decompose the eight-fold difference in trading activity into differences in bet size and bet arrival rate. The model of invariant bet frequency assumes that there are still four bets, but these bets are eight times larger \( (\gamma = 4, \tilde{Q} = 8 \cdot 250,000) \). The model of invariant bet size assumes that bets are still of 250,000 shares, but they are eight times more frequent \( (\gamma = 8 \cdot 4, Q = 250,000) \). Market microstructure invariance implies that both bet arrival rate and bet size increase. Since bet arrival rate increases twice as fast as bet size, bet arrival rate increases by a factor of four and bet size increases by a factor of two \( (\gamma = 8^{2/3} \cdot 4, \tilde{Q} = 8^{1/3} \cdot 250,000) \). Note that the bet size in business time remains the same across two games. While the number of shares traded doubles, the standard deviation of gains between bet arrivals halves, because time interval between bet arrivals decreases by a factor of four. In this sense, both trading games look the same, but one game is being played four times faster than the other.
Alternative Models and Market Depth  In order to generate predictions for market depth from the alternative models, it is useful first to make a connection between market impact costs and returns volatility. We continue to assume that market impact is linear in bet size. Since each of $\gamma$ bets generates market impact $\lambda \cdot \tilde{Q}$, the variance of the market impact of $\gamma$ bets is equal to $\gamma \cdot \lambda^2 \cdot E\{\tilde{Q}^2\}$ per day. The variance of price changes is $\sigma^2 \cdot P^2$ per day. If we assume that a fraction $\psi^2$ of price variance results from the price impact of bets, we find the following relation between the market-impact costs of bets and price volatility:

$$\psi^2 \cdot \sigma^2 \cdot P^2 = \gamma \cdot \lambda^2 \cdot E\{\tilde{Q}^2\}.$$  \hspace{0.5cm} (20)

If prices change as a result of public announcements that incorporate information into prices without trading, we tend to have $\psi < 1$. If the market impact of bets has a transitory component, we tend to have $\psi > 1$. The continuous model of Kyle (1985), where all price changes result from trading and all price impact is permanent, is consistent with $\psi = 1$.

It is easy to derive the connection between the two variables $\psi$ and $C_L$ from equations (9), (13) and (20):

$$C_L = \frac{1}{2} \cdot \psi \cdot \left[ \frac{E\{\tilde{B}^2\}}{\gamma} \right]^{1/2}.$$  \hspace{0.5cm} (21)

Under the assumptions of market microstructure invariance ($\alpha = 1/3$), the invariance of $\psi$ is equivalent to the invariance of the expected market cost of a bet $C_L$. Under alternative models, $C_L$ varies across assets and across time. In order to make predictions concerning market impact in alternative models, we need therefore to rely on another invariant rather than $C_L$ and conjecture, for example, that $\psi$ is invariant across assets and across time.

The variable $\psi$ is an intuitive market impact invariant which makes it possible to derive a simple formula for market depth applicable to both our model of microstructure invariance as well as the alternative models. Using equation (20), the dollar price impact $\lambda$ of trading one share of stock is given by

$$\lambda = \frac{\psi \cdot \sigma \cdot P}{\gamma^{1/2} \cdot [E\{\tilde{Q}^2\}]^{1/2}}.$$  \hspace{0.5cm} (22)

Note that the derivation of equation (22) does not depend on an assumed value for $\alpha$; it is valid for the model of microstructure invariance and alternative models. Equations (13) through (16) provide different ways of calculating the standard deviation of order flow imbalances $\gamma^{1/2} \cdot [E\{\tilde{Q}^2\}]^{1/2}$ in the denominator for different assumptions about the parameter $\alpha$. In other words, the three models differ only in how they infer a measure of order flow imbalances from price, volume, and volatility.

Using equations (13), (14) and (16), we find the relation between the standard deviation of order imbalances and trading volume,

$$\gamma^{1/2} \cdot [E\{\tilde{Q}^2\}]^{1/2} = \gamma^{\frac{1+\alpha}{2-2\alpha}} \cdot \left[ \frac{\zeta}{C^*} \right]^{\frac{1+\alpha}{2}} \cdot \left[ E\{\tilde{I}^2\} \right]^{1/2} \cdot \left[ \frac{W}{W^*} \right]^{\frac{\alpha+1}{2}} \cdot V,$$  \hspace{0.5cm} (23)
This equation leads to the empirical hypothesis that the standard deviation of order imbalances $\gamma^{1/2} \cdot [E\{\tilde{Q}^2\}]^{1/2}$ is equal to a fraction of trading volume $V$ proportional to $W^{(\alpha-1)/2}$, where the proportionality constant $\gamma = \frac{\zeta}{\zeta^*} \cdot \frac{1}{(1+\alpha)/2} \cdot \frac{W}{E\{I^2\}}^{1/2}$. $W^{(\alpha-1)/2}$ does not vary across assets and across time (assuming $\zeta = \zeta^*$). The three models make different assumptions about the relationship between “trading volume” and “order imbalances,” which further lead to different predictions concerning transaction costs, because of a link between volatility and the standard deviation of order imbalances. Microstructure invariance ($\alpha = 1/3$), for example, implies that the ratio of the standard deviation of order imbalances to trading volume is proportional to $W^{1/3}$. This implication follows naturally from the assumption that trading games are the same across assets, except for the speed with which they are being played.

Let $P^*$, $\sigma^*$, and $\lambda^*$ denote share price, daily volatility, and market impact for a benchmark stock which conforms to the invariance equation (20). Dividing equation (22) for some stock by equation (22) for the benchmark stock and plugging equation (23), we obtain the percentage expected market-impact cost of executing $|Q|$ shares,

$$\frac{\lambda}{P} \cdot |Q| = \frac{\lambda^* \cdot V^*}{P^*} \cdot \left[ \frac{\psi}{\psi^*} \right] \cdot \left[ \frac{\zeta}{\zeta^*} \right]^{(1+\alpha)/2} \cdot \left[ \frac{W}{W^*} \right]^{1-\alpha} \cdot \left[ \frac{\sigma}{\sigma^*} \right] \cdot \frac{|Q|}{V}. \quad (24)$$

This equation leads to the empirical hypothesis that the expected market impact cost of trading a fixed percentage of daily volume $V$ accounts for a fraction of daily volatility $\sigma$ proportional to $W^{(\alpha-1)/2}$, where the proportionality constant $\left[ \lambda \cdot V^*/P^* \right] \cdot \left[ \psi/\psi^* \right] \cdot \left[ \zeta/\zeta^* \right]^{(1+\alpha)/2} \cdot W^{(\alpha-1)/2}$ does not vary across stocks or across time (assuming $\psi = \psi^*$ and $\zeta = \zeta^*$).

Equation (24) shows why alternative models capture the thinking behind the current literature. In the model of invariant bet frequency, the assumption $\alpha = 1$ implies that market impact $\lambda/P^2$ is proportional to a ratio of daily volatility $\sigma$ and daily dollar trading volume $V \cdot P$, as in Amihud (2002). In the model of invariant bet size, the assumption $\alpha = 0$ implies that market impact is inversely proportional to the square root of daily dollar trading volume $V^{1/2} \cdot P^{1/2}$, as in the empirically motivated specification of price impact in Hasbrouck (2009). A difference with Hasbrouck (2009) is that the daily volatility $\sigma$ enters our formula in a non-linear manner, while it is absent from his specification. To see why these claims are true, one needs to express trading activity in equation (24) as the product of dollar volume and volatility. Similar thinking underlies specifications in Cooper, Groth, and Avera (1985) and Gabaix et al. (2006).

Equation (24) also shows why the alternative models conform to the way in which traders think about trading costs. In the model of invariant bet frequency, the market impact of a given fraction of daily volume $\lambda/P \cdot V$ is a constant fraction of volatility $\sigma$. This thinking is consistent with a practical rule of thumb of executing orders at a rate which keeps one’s participation rate in the market below a given threshold, for example, executing not more than one percent of trading volume.

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Alternative Models and Bid-Ask Spread  In order to generate predictions for bid-ask spread from the alternative models, we need to introduce a bid-ask spread invariant. Indeed, since the market impact cost $C_L$ is not constant in alternative models, the spread cost $C_K$ is unlikely to be constant either. To make a connection between market impact costs and bid-ask spread costs, we assume that the bid-ask spread cost of a bet is some fraction $\phi$ of expected market-impact costs, i.e.,

$$C_K = \phi \cdot C_L. \tag{25}$$

The relationship between the two variables $\phi$ and $C_K$ can be easily derived from (21) and (25),

$$C_K = \frac{1}{2} \cdot \phi \cdot \psi \cdot \left[ E\left\{ \tilde{B}^2 \right\} \right]^{1/2}. \tag{26}$$

Under the assumptions of trading game invariance and market impact invariance, the invariance of $\phi$ is equivalent to the invariance of the bid-ask spread cost of a bet $C_K$. Under the alternative models, $C_K$ varies across assets and across time. In order to make predictions concerning spread in alternative models, we conjecture that $\phi$ is constant across assets and across time. The variable $\phi$ is an intuitive bid-ask spread invariant that allows us to make predictions concerning bid-ask spread in the context of both our model and alternatives.

Using equations (9), (11), and (25) we express the bid-ask spread $\kappa$, measured in dollars per share, as

$$\kappa = \frac{\phi \cdot \lambda \cdot E\{\tilde{Q}^2\}}{E\{|\tilde{Q}|\}}. \tag{27}$$

Let $\kappa^*$ denote the bid-ask spread for a hypothetical benchmark stock which conforms to the invariance equation (25). Dividing equation (27) for the benchmark stock by equation (27) for another stock and plugging equations (16) and (24), we obtain the percentage bid-ask spread cost,

$$\frac{\kappa}{P} = \frac{\kappa^*}{P^*} \cdot \left[ \frac{\phi}{\phi^*} \right] \cdot \left[ \frac{\psi}{\psi^*} \right] \cdot \left[ \frac{\zeta}{\zeta^*} \right]^{1/2} \cdot \left[ \frac{W}{W^*} \right]^{\frac{(1-\alpha)}{2}} \cdot \left[ \frac{\sigma}{\sigma^*} \right]. \tag{28}$$

This equation leads to the empirical hypothesis that the bid-ask spread cost accounts for a fraction of daily volatility $\sigma$ proportional to $W^{-\frac{(1-\alpha)}{2}}$, where the proportionality constant $[\kappa^*/P^*] \cdot [\phi/\phi^*] \cdot [\psi/\psi^*] \cdot [\zeta/\zeta^*]^{(1-\alpha)/2}$ does not vary across assets and across time (assuming $\zeta = \zeta^*$, $\psi = \psi^*$ and $\phi = \phi^*$).

Summary of Predictions. Under the identifying assumptions $\zeta = \zeta^*$, $\psi = \psi^*$ and $\phi = \phi^*$, equations (16), (24), and (28) make precise nested predictions about how bet size (as a fraction of average daily volume), market-impact costs (controlling for trade size and volatility) and bid-ask spread costs (controlling for volatility) vary with the level of trading activity as a function of the single “deep” structural parameter $\alpha$. Market microstructure invariance predicts that $\alpha = 1/3$ and the alternative models predict that $\alpha = 0$ and $\alpha = 1$, respectively.
4 Calibration and Its Implications

In the companion empirical paper Kyle and Obizhaeva (2011), we use a database of portfolio transitions to estimate the parameter $\alpha$ in three different ways, using each of the three equations (16), (24), and (28). We do this in two steps. First, under the identifying assumption that portfolio transition orders are proportional to bets, we use the size of portfolio transition orders to test whether the predicted exponent $-(1 - \alpha)$ of trading activity $W$ in equation (16) is consistent with the prediction of market microstructure invariance that $\alpha = 1/3$. Second, using implementation shortfall to estimate trading costs as the sum of market impact costs and bid-ask spread costs, we test whether the prediction from the impact equation (24) that the exponent of trading activity $W$ is $(1 - \alpha)/2$ and the prediction from spread equation (28) that the exponent of trading activity $W$ is $-(1 - \alpha)/2$ are consistent with the implications of market impact invariance and bid-ask spread invariance that $\alpha = 1/3$.

We find that while not in perfect agreement with the predictions, the estimated exponents of -0.63, 0.33, and -0.39 are remarkably close to the predicted values of $-2/3$, $1/3$, and $-1/3$, respectively.

By imposing the predicted values of these three exponents, the companion paper estimates values of four parameters characterizing invariant constants. These parameter estimates, combined with identifying assumptions discussed below, make it possible for us to describe markets for financial assets.

4.1 Estimates from Portfolio Transitions

Our companion empirical paper estimates four parameters relevant for calibration purposes: two parameters describing the shape of the distribution of order size and two parameters describing market impact and bid-ask spread costs, respectively.

For ease of interpretation, the results are scaled to have a simple intuitive interpretation for a hypothetical benchmark stock with expected daily trading volume of $40$ million per day and expected volatility of 2% per day, i.e. with trading activity $W^* = 40 \cdot 10^6 \cdot 0.02$.

In our companion empirical paper, we find that the size of portfolio transition orders $|\tilde{X}|$ has a distribution very close in shape to a log-normal, such that $\ln(|\tilde{X}|/V)$ has a normal distribution with estimated mean $-5.69 - \frac{2}{3} \ln(W/W^*)$ and estimated variance of 2.50, i.e.,

$$\ln \left[ \frac{|\tilde{X}|}{V} \right] \approx -5.69 - \frac{2}{3} \ln \left[ \frac{W}{(0.02)(40)(10^6)} \right] + \sqrt{2.50} \cdot \tilde{Z}, \quad (29)$$

where $\tilde{Z} \equiv N(0,1)$ is the standard normal random variable. The two estimated parameters $-5.69$ (with standard error 0.018) and 2.50 describe the mean and variance of this distribution. For calibration purposes, the exponent of $W/W^*$ were fixed at the value $-2/3$ implied by microstructure invariance.
Under the maintained assumption that expected implementation shortfall is the sum of linear market impact and proportional bid-ask spread costs, the companion empirical paper uses a non-linear regression to estimate the expected cost $C(X)$, in basis points, of executing a portfolio transition order of $X$ shares as

$$C(X) = 2.89 \cdot \left( \frac{W}{(0.02)(40)(10^6)} \right)^{1/3} \cdot \frac{X}{(0.02) \cdot V} + 7.91 \cdot \left( \frac{W}{(0.02)(40)(10^6)} \right)^{-1/3} \cdot \frac{\sigma}{0.02} \cdot \frac{X}{(0.01) \cdot V}$$

In this formula, the expected order execution cost $C(X)$ is the sum of a market impact cost term consistent with equation (24) and a bid-ask spread cost term consistent with equation (28). The two estimated parameters $\bar{\lambda}/2 = 2.89$ (with standard error 0.195) and $\bar{\kappa}/2 = 7.91$ (with standard error 0.689) are scaled so that for the benchmark stock, the estimated market impact cost $C(X)$ of trading one percent of expected daily volume is sum of 2.89 basis points of impact costs and 7.91 basis points of bid-ask spread costs. For calibration purposes, the exponent of $W/W^*$ were fixed at the value $1/3$ and $-1/3$ in the market impact and spread terms, respectively, as implied by the microstructure invariance.

Note that while the two parameters describing the shape of the distribution of bet sizes are estimated with great accuracy, the two parameters describing impact and spread costs are estimated with less accuracy. As a result, conditional on additional identifying assumptions made below, the point estimates of calibrated values based on transactions cost parameters provide approximations with economically significant margin of error.

### 4.2 Implications For the Trading Process

In order to use estimates from portfolio transition orders to calibrate parameters describing the trading process for stocks, we make the identifying assumption that portfolio transition orders $\tilde{X}$ are proportional in size to bets $|\tilde{Q}|$, with the same proportionality constant $\theta$ across all stocks, i.e., $|\tilde{Q}| \approx \theta \cdot X$. Many results in this section depend on our two identifying assumptions about the values of volume deflator $\zeta$ and portfolio transition order size multiplier $\theta$. These parameters are not estimated in our paper; their calibration is an important question for future research. As a baseline case, we assume that a specialist intermediates all bets without involvement of other intermediaries ($\zeta = 2$) and portfolio transition orders are similar in size to other bets ($\theta = 1$).

Equation (29) implies a simple formula for the distribution of bet size $|\tilde{Q}|$ as a fraction of average daily volume $V$, expressed as a function of expected daily dollar trading volume $P \cdot V$ and expected daily return volatility $\sigma$:

$$\frac{|\tilde{Q}|}{V} \approx 34 \cdot 10^{-4} \cdot \theta \cdot \left[ \frac{V \cdot P \cdot \sigma}{(0.02)(40)(10^6)} \right]^{-2/3} \cdot e^{v_{2.56} Z}, \quad (31)$$

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In this equation, $\tilde{Z} \equiv N(0, 1)$ is a standard normal random variable. For the benchmark security, plugging $\tilde{Z} = 0$ into equation (31) implies that the median order size as a fraction of expected daily volume is equal to $\theta \cdot \exp(-5.69)$. For the baseline case ($\theta = 1$), this implies that the median order size for the benchmark stock is 0.34% of expected daily volume.

From equations (1) and (31), the expected number of bets per day $\gamma$ is given by

$$
\gamma = 85 \cdot \frac{\zeta}{2}^{-1} \cdot \theta^{-1} \cdot \left[ \frac{V \cdot P \cdot \sigma}{(0.02)(40)(10^6)} \right]^{2/3}.
$$

(32)

For the benchmark stock, the number of independent bets placed per day is $85 \cdot \frac{\zeta}{2}^{-1} \cdot \theta^{-1}$, or 85 bets per day for the baseline case ($\theta = 1$ and $\zeta = 2$).

The implied average size of a bet is about $472,000 \cdot \theta$ (or 11,800 $\cdot \theta$ shares). The median size of a bet is about $136,000 \cdot \theta$ (3,400 $\cdot \theta$ shares).

Equations (32) and (31) show how to extrapolate our estimates to any other security given its share price $P$, share volume $V$, and volatility $\sigma$.

### 4.3 The Contribution of Large Bets to Volume and Volatility

The estimate that bets have a log-normal distribution with variance of 2.50 implies that a large fraction of trading volume comes from large bets, and a surprisingly large fraction of returns variance comes from unusually large bets.

Let $\eta(z)$ and $N(z)$ denote the PDF and CDF of a standardized normal distribution, respectively. Define $F(\tilde{z}, m)$ by $F(\tilde{z}, m) = \int_{-\infty}^{\tilde{z}} \exp(m \cdot \sqrt{2.50} \cdot z) \cdot \eta(z) \cdot dz$. It is easy to show that $F(\tilde{z}, m) = \exp(m^2 \cdot 2.50) \cdot N(\tilde{z} - m \cdot \sqrt{2.50})$. It is straightforward to show that the fraction of the $m$th moment of order size arising from bets greater than $\tilde{z}$ standard deviations above the log-mean is given by

$$
\frac{F(\tilde{z}, m)}{F(-\infty, m)} = 1 - N(\tilde{z} - m \cdot \sqrt{2.50}).
$$

(33)

Bets larger than $\tilde{z}$ standard deviations above the log-mean (median) generate a fraction of total trading volume given by $1 - N(\tilde{z} - \sqrt{2.50})$, obtained by plugging $m = 1$ into equation (33). This implies, for example, that 50% of trading volume is generated by the largest 5.71% of bets, and 94.29% of trading volume is generated by bets larger than the 50th percentile.

Assuming bets generate permanent price impact proportional to their sizes, the contribution of a bet to price volatility is proportional to the squared size of the bet. Thus, bets larger than $\tilde{z}$ standard deviations above the log-mean (median) bet size contribute a fraction of total volatility given by $1 - N(\tilde{z} - 2 \cdot \sqrt{2.50})$, obtained by plugging $m = 2$ into equation (33). Since $2 \cdot \sqrt{2.50} = 3.16$, we infer that 50% of returns variance is generated by the 0.08% of bets larger than 3.16 standard deviations above the log-mean bet size, and 99.92% of returns variance is generated by bets greater in
size than the 50th percentile. Since the benchmark stock has about 85 bets per day, about 50% of the variance of returns in the benchmark stock is generated by large bets arriving on average once every 14 trading days. Orders that are 4 standard deviations larger than the mean—expected to arrive about once every 355 trading days for the benchmark stock—generate about 20% of returns variance over that period.

Actual trading volume and returns volatility fluctuate over time, with more volume and volatility on days when exceptionally large bets arrive in the market. For example, since rare large bets account for a significant percentage of returns variance, rare large bets may generate stochastic volatility whose properties depend on the speed with which the rare large bets are executed.

The fact that returns volatility is dominated by large bets suggests investigating in future research the extent to which large traders smooth out trades over time to avoid front-running and the extent to which rapid execution of large bets generates stochastic volatility as well as fat tails in the distribution of trading volume and returns.

4.4 Implications For Transaction Costs

The companion empirical paper provides a simple, practical formula for calculation of expected transaction costs as a function of observable dollar trading volume and volatility. Market microstructure invariance describes how transaction costs vary across assets with different levels of trading activity. A simple formula (30) for expected costs $C(X)$ shows how to extrapolate the estimated transaction costs $\hat{\lambda}/2 = 2.89$ and $\bar{K}/2 = 7.91$ basis points for the benchmark stock to any other security. Instead of using our concept of trading activity $W = P \cdot V \cdot \sigma$, equation (30) can be expressed equivalently in terms of dollar volume $P \cdot V$ and volatility $\sigma$:

$$C(X) = 2.89 \cdot \left(\frac{P \cdot V}{(40)(10^6)}\right)^{1/3} \cdot \left(\frac{\sigma}{0.02}\right)^{4/3} \cdot X \cdot \frac{1}{(0.01) \cdot V} + 7.91 \cdot \left(\frac{P \cdot V}{(40)(10^6)}\right)^{-1/3} \cdot \left(\frac{\sigma}{0.02}\right)^{2/3}$$

Note that in equation (34), the exponent of dollar volume in the market impact term is 1/3 while the exponent of volatility is 4/3. The effects of given percentage variations in volatility on market impact costs are four times greater than the effect of the same given percentage variation in dollar volume. Note also that the exponent of volatility in the market impact term is 4/3, while the exponent of volatility in the bid-ask spread term is only 2/3.

Suppose that in a time of market stress, volatility increases while dollar volume remains constant. Equation (34) then implies that if spread costs of a bet of given dollar size double as a result of increased volatility, then market impact costs increase by a factor of four. This is consistent with conventional trader wisdom that at times when increased market volatility is associated with increased bid-ask spreads, the costs of executing large trades increase much more than the costs of executing small trades.
By providing a simple formula for estimating transactions costs using readily available data on volume and volatility, equation (34) allows portfolio managers to address important questions concerning management of transactions costs. For example, given a trading strategy with known turnover demands, asset managers can estimate how fast portfolio performance decreases as the trading strategy is scaled up in size.

The implied values of transactions costs seem reasonable and consistent with typical views of market participants. For large institutional bets, market impact costs are much more important than bid-ask spread costs; for small retail bets, by contrast, the bid-ask spread cost is more important than the market impact cost. This point can be easily illustrated by comparing the expected execution costs of the average bet with execution costs for the median bet. It is a property of the log-normal distribution that the average bet is larger than the median bet by a number of standard deviations equal to half the variance. Given an estimated variance of 2.50, this corresponds to 1.25 standard deviations, equivalent to a factor of 3.57 (= \(e^{(-0.5 \cdot 2.50)}\)). Based on equation (30), the half-spread cost of the average bet is 3.57 times greater than the half-spread cost of the median bet, while the market-impact cost is 12.75 (\(= 3.57^2\)) times greater than market-impact costs of the median bet.

When comparing execution quality across brokers specializing in stocks with different levels of trading activity, performance metrics should take account of nonlinearities documented in our paper. Also, when executing basket trades, it may be inappropriate to assign the same number of basis points of transactions to each stock in a basket. Instead, transaction cost attribution to individual securities in a basket should take account of the dollar trade size, volatility, and dollar volume of the stock.

Our result in equation (34)—that the expected market impact cost of an order of given fraction of average daily volume is proportional to \((PV)^{1/3} \cdot \sigma^{4/3}\)—has an interesting relationship to the concept of market “temperature” in Derman (2002). Derman defines \(\nu\) as the arrival rate of trading opportunities for speculators, similar enough to our definition of \(\gamma\) as the arrival rate of bets that we will assume \(\nu = \gamma\). In our notation, Derman defines “temperature” \(\chi\) as \(\chi = \sigma \cdot \gamma^{1/2}\). Derman’s concept of temperature is different from our concept of trading activity in that temperature does not take into account the size of bets or the size of order imbalances. In fact, we can quantify this point precisely. Assuming the shape of the distribution of shares in a bet \(\tilde{Q}\) does not vary across stocks, then the standard deviation of order imbalances \(\sigma_U = \gamma^{1/2} \cdot E\{\tilde{Q}^2\}^{1/2}\) is proportional to \(\gamma^{1/2} \cdot E\{|\tilde{Q}|\}\). Writing the definition of trading activity as \(W = \gamma \cdot P \cdot \sigma \cdot E\{|\tilde{Q}|\}\), it is easy to show that trading activity \(W\) is proportional to the product of the dollar standard deviation of order imbalances \(P\sigma_U\) and market temperature \(\chi\), i.e., \(P\sigma_U \cdot \chi \cdot W^{-1}\) is a constant; this result does not depend on microstructure invariance assumptions. Microstructure invariance implies that \(\gamma\) is proportional to \(W^{2/3}\), from which it follows that temperature is proportional to \((PV)^{1/3} \cdot \sigma^{4/3}\). Thus, microstructure invariance implies that Derman’s definition of temperature is proportional to the market impact costs (in basis points) of an order.
of a given fraction of average daily volume (equation (34)).

4.5 Implications for Order Imbalances and Kyle (1985)

Since bets are assumed to be approximately independently distributed, the variance of order flow imbalances can be calculated as the sum of the variances of the individual bets. Equations (31) and (32) imply a specific structural relationship between the standard deviation of order imbalances $\gamma = 2 \cdot \left[ E\{\tilde{Q}^2\} \right]^{1/2}$ and volume $V$,

$$\gamma^{1/2} \cdot [E\{\tilde{Q}^2\}]^{1/2} = 0.38 \cdot \left[ \frac{\zeta}{2} \right]^{-1/2} \cdot \theta^{1/2} \cdot \left[ \frac{V \cdot P \cdot \sigma}{(0.02)(40)(10^6)} \right]^{-1/3} \cdot V. \quad (35)$$

In the baseline scenario ($\theta = 1$ and $\zeta = 2$), the standard deviation of daily order imbalances is 38% of daily trading volume for the benchmark stock. For other financial assets, the standard deviation of order imbalances is proportional to trading volume scaled by $W^{-1/3}$.

It is this structural relationship between the unobservable standard deviation of order imbalances and observable trading volume that allows us to generate testable predictions based on the model of Kyle (1985) and other microstructure models, where order flow imbalances move prices with linear price impact and the impact parameter depends on the standard deviation of order flow imbalances.

In Kyle (1985), for example, trading takes place over an arbitrary period of time called a trading day. For the purpose of using this model to measure market depth empirically, however, there is no a priori reason to assume that this trading day is literally one calendar day. Furthermore, the length of the trading day may vary across assets, being a much shorter period of time for more active stocks. If Kyle (1985) describes trading games in both inactive and active stocks and the length of trading day varies across assets in a manner consistent with microstructure invariance (i.e., inversely proportional to $\gamma$), then the standard deviation of order imbalances—per calendar day—is related to trading volume by equation (35).

Equation (22), $\lambda = \psi \cdot \sigma \cdot \frac{P}{[\gamma^{1/2} \cdot E\{\tilde{Q}^2\}]^{1/2}}$, is very similar to the formula $\lambda = \sigma_V / \sigma_U$ from the continuous model of Kyle (1985).

In the numerator of $\lambda = \sigma_V / \sigma_U$, the parameter $\sigma_V$ denotes the dollar standard deviation of an informed trader’s private information; in the continuous equilibrium, this variable also equals the dollar standard deviation of price changes over one trading day, assuming that market makers make markets semi-strong efficient by correctly anticipating the adverse selection in the order flow. In equation (22), the numerator $\psi \cdot \sigma \cdot P$ also measures the dollar standard deviation of price changes resulting from the price impact of trading over one calendar day. Since the formula for $\lambda$ in Kyle (1985) does not depend on the time horizon, which cancels out from both numerator and denominator, we can think of both $\sigma_V$ and $\sigma_U$ as calculated per calendar day. Thus, it is reasonable to assume $\sigma_V = \psi \cdot \sigma \cdot P$ for the purpose of testing Kyle (1985) empirically.
In the denominator of $\lambda = \sigma_V / \sigma_U$, the parameter $\sigma_U$ denotes the standard deviation of the net quantity traded by noise traders in one trading day; in the continuous equilibrium, this variable also equals the standard deviation of the combined order flow from both informed and noise traders. The informed trader smooths out his trading and therefore his trading does not contribute to order flow variance. In equation (22), the denominator measures the standard deviation of the net quantity traded by all bets placed in one calendar day. Therefore, it is appropriate to assume $\sigma_U = \left[ \gamma^{1/2} \cdot E\{\tilde{Q}^2\} \right]^{1/2}$ for the purpose of testing Kyle (1985) empirically. To conform with the intuition of Kyle (1985), these bets should contain a mix of informed trading and noise trading.

Using dollar volatility per calendar day as a proxy for $\sigma_V$ as well as trading volume $V$ scaled by $W^{-1/3}$ and a constant from (35) as an empirical proxy for $\sigma_U$, it is easy to show that the formula $\lambda = \sigma_V / \sigma_U$ implies that market impact is proportional to $W^{1/3}$, as in our formula (34).

Microstructure invariance therefore does not contradict theoretical microstructure models. Rather it compliments them, making it possible to test these models empirically in a manner that capture cross-sectional variation in volume and volatility across stocks with different levels of trading activity.

4.6 Log-Normality of Bet Size and the Distribution of Price Changes

Our paper provides a different perspective on the “time change” literature which goes back to Mandelbrot and Taylor (1967) and Clark (1973). The time change literature begins with the empirical observation that the distribution of price changes has a kurtosis greater than a normal distribution, i.e., a sharper peak near zero and and fatter tails. Daily price changes result from normally and independently distributed small price increments in response to individual trades executed during the day. The main idea of “time-change” literature is that fat tails can be explained by business time operating at speeds different from calendar time, so that the rate at which trades are executed in the market varies from day to day. Both papers advance the idea that the distribution of daily price changes is subordinate to a normal distribution with a time clock potentially linked to trading volume.

Mandelbrot and Taylor (1967) begin with the intuition that the distribution of price changes is a stable distribution, i.e., a distribution such that a linear combination of two independent random variables has the same shape, up to location and scale parameters. Stable distributions are a subset of infinitely divisible distributions, i.e., distributions such that, for any $n$, there exist $n$ independent identically distributed random variables whose sum has that distribution. Mandelbrot and Taylor (1967) use characteristic functions to derive the set of distributions consistent with the distributions being stable. If price changes follow a stable distribution, then the fact that price changes have more kurtosis than the normal implies that price changes
have infinite variance, because the only stable distribution with finite variance is the normal distribution. Stable distributions that are non-normal are often called stable Pareto distributions or power-law distributions.

Following Mandelbrot and Taylor (1967), the econophysics literature estimates different power-laws for the probability distributions of different variables. Gopikrishnan et al. (1998) find a Pareto exponent of 3 for stock price fluctuations, \( P(|R| > x) \sim x^{-3} \). Plerou et al. (2000) report the same Pareto exponent for the number of transactions over a given period of time and the variance of price changes between subsequent transactions, \( P(N > x) \sim x^{-3} \) and \( P(|r|^2 > x) \sim x^{-3} \). They conclude that—if market impact is linear in trade size and trades are independently and identically distributed—price fluctuations can be described as a diffusion with a random diffusion constant linked to the changing variance of price increments between subsequent transactions \(|r|^2\), but not the number of trades \(N\). Gabaix et al. (2006) also notice that empirical price fluctuations would be consistent with the square-root price impact of independently and identically distributed trades. Alternatively, Bouchaud, Farmer, and Lillo (2009) emphasize that order flow is a highly autocorrelated long-memory process and suggest their own model for how markets digest changes in demand and supply.

Clark (1973) differs from the power-law literature. Clark (1973) proposes that the number of small price changes per day follows a log-normal distribution and—under the assumption that individual price increments are identically distributed—it is proportional to trading volume. These approaches generate a price process with a finite variance for returns.

The approach of market microstructure invariance is different from the previous literature. Microstructure invariance is based on the idea that the number of bets arriving in the market is approximately generated by a Poisson process, which implies that the number of bets per calendar day has a Poisson distribution. The price changes resulting from bets are distributed symmetrically about zero and, assuming linear impact, have an unsigned size which follows a log-normal distribution with log-variance estimated to be 2.50.

In microstructure invariance, the excess kurtosis in daily price changes is positive both because the number of bets is random and because the unsigned bet size is log-normally distributed (with signed bet size distributed symmetrically about zero). Given the distribution of bet sizes in equation (31), microstructure invariance implies that the excess kurtosis of one bet has the enormous value of \( \exp(10) \) or about 22,000. Given that bets are generated by a Poisson process with an expected number of bets satisfying equation (32), it is easy to derive formulas for excess kurtosis for different horizons. Note that microstructure invariance predicts that the level of excess kurtosis decreases with the horizon, i.e., kurtosis gradually decreases from high levels for daily and weekly returns to low levels for monthly and annual returns, gradually converging to zero, as the distribution of price changes converges to a normal when the horizon goes to infinity (assuming underlying volume and volatility do not change).
Examining kurtosis of price changes at different horizons might help to differentiate between the hypothesis of microstructure invariance and the power-law distributions, which should have the same shape regardless of horizons. Calculating kurtosis for different horizons, however, does not make sense in the context of Clark (1973). The log-normal distribution is neither stable nor infinitely divisible; the sum of random variables with independent log-normal distributions is not log-normal. Thus, if daily price changes can be described by Clark’s hypothesis, neither a half-day price changes nor weekly price changes will be described by the same hypothesis.

Although these “time change” models are based on the intuition that trading volume is related to the manner in which time is transformed, the transformation takes place differently in different models. Clark (1973) models time in volume units, with each unit of volume generating a small price change. This is similar in spirit to the model of invariant bet size (Hasbrouck). It is also similar to Gabaix et al. (2006). Mandelbrot and Taylor (1967) imagine large and small orders arriving in the market. In this respect, their approach seems similar to ours, even though they model the number and size of orders in a different manner. Note that the power-law literature is based on the concept of a transaction, not a bet. To the extent that the power-law literature infers transactions from TAQ data, the approach of this literature is different from ours, because a bet is likely to generate many transactions, as a result of which the actual order flow has economically significant positive autocorrelation.

Whether order size follows a log-normal distribution or a power law is an interesting question for future research. Log-normality implies that the right tail of a plot of logs or rank against quantiles from an empirical distribution should look concave and quadratic. Power laws imply the plot should look linear. Our companion empirical paper finds some visual evidence of linearity in the far right tails. It is difficult, however, to distinguish a distribution with a power law in the right tail from a log-normal distribution with a large variance.

The log-normality of bet size is strikingly different from typical assumptions of microstructure models, where innovations in order flow from noise traders is distributed as a normal, not a log-normal or power law. Although normal random variables are a convenient modeling device because they allow conditional expectations to be linear functions of underlying jointly normally distributed variables, their implications for the size of price fluctuations are very different.

4.7 Liquidity and Market Velocity

In this section, we examine two different definitions of “liquidity” based on either (1) the cost of converting an asset to cash or (2) the cost of transferring a risk. We show that both of these measures are related to market velocity, or the speed with which bets are placed and money changes hands, measured by bet arrival rate $\gamma$.

Let $1/L_5$ denote a liquidity index measuring the expected cost of converting an asset to cash as a fraction of the dollar value of the asset (e.g., in basis points). Let
$1/L_\alpha$ denote a liquidity index measuring the expected cost of transferring a risk of
the sizes that are exchanged in the market (e.g., similarly to a Sharpe ratio, in units
of risk). We think of trading costs as the reciprocal of liquidity, so increasing liquidity
$L_S$ and $L_\alpha$ implies lower trading costs.

Our first measure of liquidity $1/L_S$ measures the dollar-volume-weighted average
transactions costs, expressed in basis points. Using equations (30) and (31), we find
that

$$
1/L_S = \left[ \frac{1960 \cdot \theta + 373}{471,838} \right] \cdot \left( \frac{W}{W^*} \right)^{-1/3} \cdot \frac{\sigma}{\sigma^*} = \left[ \frac{1960 \cdot \theta + 373}{471,838} \right] \cdot \left( \frac{P \cdot V}{4 \cdot 10^6} \right)^{-1/3} \cdot \left( \frac{\sigma}{0.02} \right)^{2/3}.
$$

(36)

Note that $L_S$ is proportional to the cube root of dollar volume per unit of returns
variance $[P \cdot V / \sigma^2]^{1/3}$. “Dollars per unit of variance” is a simple and intuitive—if ad
hoc—way to proxy for liquidity. The invariance hypothesis implies that taking the
cube root generates a sensible index for measuring liquidity based on the cost of
converting an asset to cash.

Our second liquidity measure $1/L_\alpha$ measures the cost of exchanging risks of the
sizes that market participants exchange in the market. It is expressed in standard
deviation units and defined as $1/L_S$ divided by $\sigma$,

$$
1/L_\alpha = \left[ \frac{1960 \cdot \theta + 373}{472,000 \cdot 0.02} \right] \cdot \left( \frac{W}{W^*} \right)^{-1/3} = \left[ \frac{1960 \cdot \theta + 373}{472,000 \cdot 0.02} \right] \cdot \left( \frac{P \cdot V \cdot \sigma}{40 \cdot 10^6 \cdot 0.02} \right)^{-1/3}.
$$

(37)

Note that $L_\alpha$ is proportional to the cube root of trading activity $[P \cdot V \cdot \sigma]^{1/3}$. The
invariance hypothesis implies that taking the cube root generates a sensible index for
measuring liquidity based on the cost of transferring risks.

Both liquidity measures $L_S$ and $L_\alpha$ can be expressed in terms of market velocity
$\gamma$. Plugging equation (32) into equations (36) and (37), we obtain

$$
1/L_S = 1/L_\alpha \cdot \sigma = \left[ \frac{1960 \cdot \theta + 373}{472,000} \right] \cdot \left[ \frac{\zeta}{2} \cdot \theta \right]^{-1/2} \cdot \left( \frac{\gamma}{85} \right)^{-1/2} \cdot \frac{\sigma}{0.02}.
$$

(38)

For a baseline case ($\zeta = 2$ and $\theta = 1$), this becomes

$$
1/L_S = 1/L_\alpha \cdot \sigma = 0.25 \cdot \left( \frac{\gamma}{85} \right)^{-1/2} \cdot \sigma.
$$

(39)

For the benchmark stock in the baseline scenario ($\zeta = 2$ and $\theta = 1$), $1/L_S^*$ is 50 basis
points and $1/L_\alpha^*$ is 25 percent of daily volatility. Note that $L_S$ is proportional to the
square root of market velocity $\gamma$ per unit of returns variance $\sigma^2$. $L_\alpha$ is proportional
to the square root of market velocity.

The measures of liquidity, $L_S$ and $L_\alpha$, are useful for different purposes. For ex-
ample, consider what happens if volatility increases, holding dollar trading volume
constant. Does liquidity increase or decrease? The value of $L_S$ decreases, implying
that increased volatility increases the costs of converting assets to cash. The value
of $L_\sigma$ increases, implying that increased volatility encourages speculation by making it less costly to transfer risks of given dollar standard deviation. These different measures of liquidity explain why some traders might say increased volatility makes market less liquid while other traders might say that increased volatility makes markets more liquid.

4.8 Calibration of Parameter Values

Finally, we use the four parameter from the companion empirical paper to calibrate the market microstructure invariants: the shape of distribution of $\tilde{I}$, values of $C_L$ and $C_K$. We also calibrate values of $\phi$ and $\psi$. All calibrated variables seem to be economically reasonable.

**Calibration of $\tilde{I}$.** Plugging $|\tilde{Q}| \approx \theta \cdot X$ into equation (8) and taking logs, we obtain the following:

$$\ln \left[ \frac{X}{V} \right] + \frac{2}{3} \ln \left[ \frac{W}{W^*} \right] \approx \ln(|\tilde{I}|) - \frac{1}{3} \ln E\{|\tilde{I}|\} - \frac{1}{3} \ln \frac{\zeta}{2} - \ln \theta - \frac{2}{3} \ln W^*. \quad (40)$$

This equation shows that, except for the differences in the means, the log-invariant $\ln(|\tilde{I}|)$ is distributed similarly to $\ln \left[ \frac{X}{V} \right] + \frac{2}{3} \ln \left[ \frac{W}{W^*} \right]$, as a normal.

The distribution of the trading game invariant $\tilde{I}$ can be therefore approximated by the product of two random variables: (1) a buy-sell indicator variable $\tilde{\delta}$ assuming values of +1 or −1 with equal probability, and (2) a log-normal random variable with log-mean $\mu_I$ and log-variance $\sigma_I^2$,

$$\tilde{I} \approx \tilde{\delta} \cdot e^{\mu_I + \sigma_I \tilde{Z}}. \quad (41)$$

To find $\mu_I$ and $\sigma_I^2$, we match the mean and the variance of the right-hand side variables in equation (40) to -5.69 and 2.50, respectively, from equation (29),

$$E\{\ln \left[ \frac{X_i}{V_i} \right] + \frac{2}{3} \ln \left[ \frac{W_i}{W^*} \right]\} = -\frac{1}{3} \ln \frac{\zeta}{2} - \ln \theta - \frac{2}{3} \ln W^* - \frac{1}{6} \sigma_I^2 + \frac{2}{3} \mu_I = -5.69, \quad (42)$$

$$\text{Var}\{\ln \left[ \frac{X_i}{V_i} \right] + \frac{2}{3} \ln \left[ \frac{W_i}{W^*} \right]\} = \sigma_I^2 = 2.50. \quad (43)$$

Equations (42) and (43) imply the following values for $\mu_I$ and $\sigma_I^2$:3

$$\mu_I = 5.6824 + \frac{1}{2} \ln(\zeta/2) + \frac{3}{2} \ln \theta, \quad (44)$$

3The number -5.69 in equation (42) and the number 5.6824 in equation (44) are related by the following arithmetics: $5.6824 = -5.69 \cdot 3/2 + \ln(0.02 \cdot 40 \cdot 10^6) + 1/4 \cdot 2.50$ It is a coincidence that they have similar values.
\[ \sigma_I^2 = 2.50 \]  
\( \text{(45)} \)

Note that from equation (44), the value of \( \mu_I \) must be adjusted for the non-bet volume inflator \( \zeta \) and the portfolio transition order multiplier \( \theta \). In a baseline case \( (\zeta = 2, \theta = 1) \), the trading game invariant \( \tilde{I} \) is distributed as,

\[ \tilde{I} \approx \tilde{\delta} \cdot e^{5.6824+\sqrt{2.50} \tilde{Z}}. \]  
\( \text{(46)} \)

**Calibration of the Expected Market Impact Cost Parameter \( C_L \).** The expected dollar cost of executing a bet, \( \mathbb{E}\{P \cdot \tilde{Q} \cdot C(\tilde{Q})\} \), is equal to the sum of dollar market impact costs \( C_L \) and dollar half-spread costs \( C_K \). From equations (30) and (31), we obtain the following estimate of the expected market-impact cost of a bet:

\[ C_L = 1,960 \cdot \theta^2. \]  
\( \text{(47)} \)

The assumption of market impact invariance implies that \( C_L \) does not depend on the level of trading activity \( W \). Furthermore, it does not depend on the parameter \( \zeta \). It does depend, however, on our identifying assumption about the factor \( \theta \) used to deflate the distribution of portfolio transition orders to match bets. If transition orders are similar to typical bets \( (\theta = 1) \), then the expected cost of a bet is equal to \$1,960. The larger are portfolio transition orders relative to other bets, the smaller is the expected market-impact cost of a bet.

**Calibration of the Expected Bid-Ask Spread Cost of a Bet \( C_K \).** From equations (30) and (31), we also obtain the estimate of the expected half-spread cost of a bet:

\[ C_K = 373. \]  
\( \text{(48)} \)

The assumption of bid-ask spread invariance implies that \( C_K \) depends neither on trading activity \( W \) nor the volume deflator \( \zeta \). Unlike the expected market impact cost of a bet, it also does not depend on \( \theta \).

The parameters \( C_L \) and \( C_K \) can be also expressed in terms of parameters \( \psi \) and \( \phi \).

**Calibration of \( \psi \).** The parameter \( \psi \) relates the aggregate price impact of bets to the total variance of returns. Using equations (41), (44), and (45), we express \( \psi \) from equation (21),

\[ \psi = 1.10 \cdot \left[ \frac{\zeta}{2} \right]^{-1/2} \cdot \theta^{1/2}. \]  
\( \text{(49)} \)

The calibrated values for \( \psi \) depend on the identification assumptions about \( \zeta \) and \( \theta \). Under the baseline scenario \( (\zeta = 2 \text{ and } \theta = 1) \), \( \psi = 1.10 \) implies that execution of bets is associated with 110% of returns variance. The fact that \( \psi \) is calibrated to be close to 1 is economically realistic. In our calibration, we have made the identifying assumption that the market impact cost of executing portfolio transition
orders is half of their actual market impact on prices. If, instead, this market impact cost is more than their actual impact—either because high frequency traders may be able to detect large bets and make their execution more expensive by trading in front of them or because execution of these orders may incur additional costs due to transitory price changes—then the parameter $\psi$ can be greater than 1. Since some fraction of volatility also results from public announcements, these effects may be even more significant. Studying the decomposition of market impact into transitory and permanent components is an interesting issue for future research.

**Calibration of $\phi$.** The parameter $\phi$ relates the expected market impact cost of a bet to the spread cost. Equations (25), (47), and (48) imply that $\phi$ is equal to,

$$\phi = 0.19 \cdot \theta^{-2}. \quad (50)$$

Under the assumption of $\theta = 1$, this equation implies that the expected market impact cost of a bet $1960 is about five times larger than the expected spread cost of a bet $373. These intuitively reasonable approximations provide an additional consistency check for our invariance hypothesis.

## 5 Conclusion

Market microstructure invariance generates the testable empirical predictions that bets arrive approximately randomly at a rate proportional to $(P \cdot V \cdot \sigma)^{2/3}$ as in equation (32), their size is distributed closely to a log-normal with the median size proportional to $(P \cdot V \cdot \sigma)^{2/3}$ and the log-variance of 2.50 as in equation (31), market impact costs of trading a given percentage of expected daily volume are proportional to $(P \cdot V)^{1/3} \cdot \sigma^{4/3}$, and bid-ask spread costs are proportional to $(P \cdot V)^{-1/3} \cdot \sigma^{2/3}$ as in equations (30) and (34). These formulas take as their only inputs daily dollar volume $P \cdot V$, and daily percentage returns volatility $\sigma$. In addition to the log-variance of 2.50, practical application of these formulas requires estimation of three scaling parameters. In terms of the benchmark stock, the companion empirical paper calibrates these three parameters as median order size equal to 0.34% of daily volume, market impact costs of trading one percent of daily volume equal to 2.91 basis points, and bid-ask spread costs equal to 7.91 basis points.

We conjecture that the predictions of market microstructure invariance can be found to hold not only in portfolio transitions data for equities, as shown in our companion empirical paper, but also in other data such as transactions in the Trades and Quotes (“TAQ”) dataset, changes in holdings recorded in 13-F filings of institutional investment managers, institutional trades reported in the Ancerno dataset, and other datasets. We conjecture that the flow of news articles may be constant in units of business time as well. The execution of very large bets may be associated with short-term stochastic volatility. Large market disturbances such as the stock market crash
of 1987, the liquidation of Jerome Kerviel’s trades for Société Générale, and the flash crash of May 6, 2010 (see Kirilenko et al. (2010)) may be the result of executing large bets.

We conjecture that market frictions such as wide tick size, minimum round lot sizes, imperfect competition among market makers, lack of pre-trade and post-trade transparency, transactions taxes, and poor electronic interfaces may result in deviations from the predictions of market microstructure invariance. In particular, such frictions may increase bid-ask spread costs. When market frictions are important, market microstructure invariance can be used as a benchmark for measuring their importance.

We conjecture that the predictions of market microstructure invariance may generalize to other markets such as bond markets, currency markets, and futures markets, as well as to other countries. Whether the market microstructure invariance hypothesis is indeed of such a general nature is an interesting issue for future research.

The three invariants \( \tilde{I} \), \( C_L \), and \( C_K \) are measured in this paper in dollars. If the invariance hypotheses are applied in an international context to markets with different currencies or different real exchange rates or applied across period of time where the price level is changing significantly, we conjecture that the three invariants should be deflated by the real wages of finance professionals in the local currency of the given market. Like fundamental constants in physics, such deflation will make our invariants dimensionless.

Although microstructure invariance is presented in this paper as an empirical hypothesis, its empirical validity should be a result of equilibrium trading in financial markets. In equilibrium, traders search for ideas. When a good idea is found, traders place bets and expect to profit from them. In comparing active markets with inactive markets, our model is broadly consistent with the conjecture that traders in different markets have the same level of skills and earn the same wages, discover ideas at the same rate (in units of their own effort), place bets at the same rate, expect to pay the same transactions costs per bet, and expect to make the same profits per bet. Although bets are larger in active markets, they are held for shorter periods of time and so risks transferred by each bet do not vary across markets. Traders who specialize in trading inactive assets may trade simultaneously more assets than those who specialize in trading active assets, because traders have to place the same number of bets per unit of time, while being constrained by different business-time clocks in different markets. Developing a more formal equilibrium model which generates predictions consistent with this informal story of market microstructure invariance requires addressing interesting modeling issues that will probably uncover important insights. Developing such a model takes us beyond the scope of this paper but is an interesting topic for future research.
References


The figure illustrates differences between market microstructure invariance and the two alternatives by comparing two hypothetical stocks. Their prices and volatilities are assumed to be the same, but trading volume of active stock (top chart) is eight times larger than trading volume of inactive stock (bottom chart). Buy orders are marked in green color, and sell orders are marked in red color. The lengths of the bars are proportional to dollar bet size. Inactive stock has four bets expected per calendar day. The model of market microstructure invariance suggests that active stock has 4 times as many bets per day, and the dollar size of each bet is twice as large, to keep volatility constant. The model of “invariant bet frequency” assumes that the difference in volume comes from the difference in bet sizes. The model of “invariant bet size” assumes that the difference in volume comes from the difference in bet frequencies.